PhD Summer School in Discrete Mathematics

Marston Conder ▪ Edward Dobson ▪ Tatsuro Ito

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Preface

This is a collection of lecture notes of the PhD Summer School in Discrete Mathematics, held from June 16 to June 21, 2013, by tradition at Rogla, Slovenia. The organization of this summer school came as a combined effort the Faculty of Mathematics, Natural Sciences and Information Technologies and the Andrej Marušič Institute at the University of Primorska, and the Centre for Discrete Mathematics at the Faculty of Education at the University of Ljubljana.

The Scientific Committee of the meeting consisted of Klavdija Kutnar, Aleksander Malnič, Dragan Marušič, Štefko Miklavič and Primož Šparl. The Organizing Committee of the meeting consisted of Ademir Hujdurović, Boštjan Frelih and Boštjan Kuzman.

The aim of this Summer School was to bring together senior researchers, junior researches and PhD students working in Algebraic Graph Theory. The summer school has consisted of three minicourses given by

- Prof. Marston Conder, University of Auckland, New Zealand,
- Prof. Edward T. Dobson, Mississippi State University, USA & UP, Slovenia, and
- Prof. Tatsuro Ito, Kanazawa University, Japan.

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Graph Symmetries

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SUMMARY

- Introduction to symmetries of graphs
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- Proof of Tutte's theorem on symmetric cubic graphs
- Use of amalgams and covers to analyse and construct examples
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1.1 Introduction to Symmetries of Graphs

Generally, an object is said to have **symmetry** if it can be *transformed in way that leaves it looking the same as it did originally.*

Automorphisms: An *automorphism* (or *symmetry*) of a simple graph $X = (V, E)$ is a permutation of the vertices of X which preserves the relation of adjacency; that is, a bijection $\pi : V \to V$ such that $\{v^\pi, w^\pi\} \in E$ if and only if $\{v, w\} \in E$.

Under composition, the automorphisms form a group, called the *automorphism group* (or *symmetry group*) of X, and this is denoted by Aut(X), or Aut X.

Examples

(a) Complete graphs and null graphs: $\mathrm{Aut}\, K_n \cong \mathrm{Aut}\, N_n \cong S_n$ for all n.

(b) Simple cycles: $\mathrm{Aut}\, C_n \cong D_n$ (dihedral group of order $2n$) for all $n \geq 3$.

(c) Simple paths: $\mathrm{Aut}\, P_n \cong S_2$ for all $n \geq 3$.

(d) Complete bipartite graphs: $\mathrm{Aut}\, K_{m,n} \cong S_m \times S_n$ when $m \neq n$, while $\mathrm{Aut}\, K_{n,n} \cong S_n \wr S_2 \cong (S_n \times S_n) \rtimes S_2$ (when $m = n$).

(e) Star graphs – see above: $\mathrm{Aut}\, K_{1,n} \cong S_n$ for all $n > 1$.

(f) Wheel graphs (cycle C_{n-1} plus nth vertex joined to all): $\mathrm{Aut}\, W_n \cong D_{n-1}$ for all $n \geq 5$.

(g) Petersen graph: $\mathrm{Aut}\, P \cong S_5$.

Exercise 1: How many automorphisms has the underlying graph (1-skeleton) of each of the five Platonic solids: the regular tetrahedron, cube, octahedron, dodecahedron and icosahedron?

Exercise 2: Find a simple graph on 6 vertices that has exactly one automorphism.

Exercise 3: Find a simple graph that has exactly three automorphisms. What is the *smallest* such graph?

Exercise 4: For large n, do 'most' graphs of order n have a large automorphism group? or just the identity automorphism?

One amazing fact about graphs and groups is **Frucht's theorem**: in 1939, Robert(o) Frucht proved that given any finite group G, there exist infinitely many connected graphs X such that Aut X is isomorphic to G. And then later, in 1949, he proved that X may be chosen to be 3-valent. There are several variants and generalisations of this, such as regular representations for graphs and digraphs (GRRs and DRRs).

The simple graphs of order n with the largest number of automorphisms are the null graph N_n and the complete graph K_n, each with automorphism group S_n (the symmetric group on n symbols). For non-null, incomplete graphs of bounded valency (vertex degree), the situation is more interesting.

In this course of lectures, we will devote quote a lot of attention to the case of regular graphs of valency 3, which are often called *cubic graphs.*

Exercise 5: For each $n \in \{4, 6, 8, 10, 12, 14, 16\}$, which 3-valent connected graph on n vertices has the largest number of automorphisms?

For fixed valency, some of the graphs with the largest number of automorphisms do not look particularly nice, or do not have other good properties (e.g. strength/stability, or suitability for broadcast networks). The 'best' graphs possess special kinds of symmetry.

Transitivity: A graph $X=(V,E)$ is said to be

- *vertex-transitive* if $\mathrm{Aut}\,X$ has a single orbit on the vertex-set V;
- *edge-transitive* if $\mathrm{Aut}\,X$ has a single orbit on the edge-set E;
- *arc-transitive* (or *symmetric*) if $\mathrm{Aut}\,X$ has a single orbit on the arc-set (that is, the set $A=\{(v,w)\mid\{v,w\}\in E\}$ of all ordered pairs of adjacent vertices);
- *distance-transitive* if $\mathrm{Aut}\,X$ has a single orbit on each of the sets $\{(v,w)\mid d(v,w)=k\}$ for $k=0,1,2,\ldots$;
- *semi-symmetric* if X is edge-transitive but not vertex-transitive;
- *half-arc-transitive* if X is vertex-transitive and edge-transitive but not arc-transitive.

Examples

(a) Complete graphs: K_n is vertex-transitive, edge-transitive, arc-transitive and distance-transitive.

(b) Simple cycles: C_n is vertex-transitive, edge-transitive, arc-transitive and distance-transitive.

(c) Complete bipartite graphs: $K_{m,n}$ is edge-transitive, but is vertex-transitive (and arc-transitive and distance-transitive) only when $m=n$.

(d) Wheel graphs: W_n is neither vertex-transitive nor edge-transitive (for $n\geq 5$).

(e) The Petersen graph is vertex-transitive, edge-transitive, arc-transitive and distance-transitive.

Note that **every vertex-transitive graph is regular** (in the sense of having all vertices of the same degree/valency), since for any two vertices v and w, there is an automorphism θ taking v to w, and then θ takes the edges incident with v to the edges incident with w.

On the other hand, not every edge-transitive graph is regular: counter-examples include all $K_{m,n}$ for $m\neq n$. But there exist graphs that are edge-transitive and regular but not vertex-transitive. One example is the smallest semi-symmetric cubic graph, called the *Gray graph* (discovered by Gray and re-discovered later by Bouwer), on 54 vertices. The smallest semi-symmetric regular graph is the *Folkman graph*, which is 4-valent on 20 vertices.

Also note that every arc-transitive connected graph without isolated vertices is both vertex-transitive and edge-transitive, but the converse does not hold. Counter-examples are half-arc-transitive. The smallest half-arc-transitive graph is the *Holt graph*, which is a 4-valent graph on 27 vertices. There are infinitely many larger examples.

Exercise 6: Let X be a k-valent graph, where k is odd (say $k=3$). Show that if X is both vertex-transitive and edge-transitive, then also X is arc-transitive. [Harder question: does the same thing always happen when k is even?]

Exercise 7: Prove that every semi-symmetric graph is bipartite.

Exercise 8: Every distance-transitive graph is arc-transitive. Can you find an arc-transitive graph that is not distance-transitive?

***s*-arcs**: An *s-arc* in a graph $X = (V, E)$ is a sequence $(v_0, v_1, \ldots, v_s)$ of vertices of X in which any two consecutive vertices are adjacent and any three consecutive vertices are distinct, that is, $\{v_{i-1}, v_i\} \in E$ for $1 \le i \le s$ and $v_{i-1} \ne v_{i+1}$ for $1 \le i < s$. The graph $X = (V, E)$ is called *s-arc-transitive* if $\mathrm{Aut}\,X$ is transitive on the set of all s-arcs in X.

Examples

(a) Simple cycles: C_n is s-arc-transitive for all $s \ge 0$, whenever $n \ge 3$.

(b) Complete graphs: K_n is 2-arc-transitive, but not 3-arc-transitive, for all $n \ge 3$.

(c) The complete bipartite graph $K_{n,n}$ is 3-arc- but not 4-arc-transitive, for all $n \ge 2$.

(d) The Petersen graph is 3-arc-transitive, but not 4-arc-transitive.

(e) The Heawood graph (the incidence graph of the projective plane of order 2) is 4-arc-transitive, but not 5-arc-transitive.

Exercise 9: For each of the five Platonic solids, what is the largest value of s such that the underlying graph (1-skeleton) is s-arc-transitive?

Exercise 10: Let X be an s-arc-transitive d-valent connected simple graph. Find a lower bound on the order of the stabiliser in $\mathrm{Aut}\,X$ of a vertex $v \in V(X)$, in terms of s and d.

Sharp transitivity ('regularity'): A graph $X = (V, E)$ is said to be

- *vertex-regular* if the action of $\mathrm{Aut}\,X$ on the vertex-set V is regular (that is, for every ordered pair (v, w) of vertices, there is a *unique* automorphism taking v to w);
- *edge-regular* if the action of $\mathrm{Aut}\,X$ on the edge-set E is regular;
- *arc-regular* if the action of $\mathrm{Aut}\,X$ on the arc-set A is regular;
- *s-arc-regular* if the action of $\mathrm{Aut}\,X$ on the set of s-arcs of X is regular.

The same terminology applies to actions of a subgroup of $\mathrm{Aut}\,X$ on X. For example, *Cayley graphs* (which will be encountered soon) are precisely the graphs that admit a vertex-regular group of automorphisms ... and possibly other automorphisms as well.

Important note: The term 'distance regular' means something quite different – a graph X is called *distance regular* if for all j and k, it has the property that for any two vertices v and w at distance j from each other, the number of vertices adjacent to w and at distance k from v is a constant (depending only on j and k, and not on v and w).

1.2 Vertex-transitive and Arc-transitive Graphs

Let X be a vertex-transitive graph, with automorphism group G, and let H be the stabiliser of any vertex v, that is, the subgroup $H = G_v = \{g \in G \mid v^g = v\}$. Let us also assume that X is not null, and hence that every vertex of X has the same positive valency.

Since G is transitive on $V = V(X)$, we may label the vertices with the right cosets of H in G such that each automorphism $g \in G$ takes the vertex labelled H to the vertex labelled Hg — that is, the action of G on $V(X)$ is given by right multiplication on the coset space $(G:H) = \{Hg : g \in G\}$.

Next, **define** $D = \{g \in G \mid v^g \text{ is adjacent to } v\} = \{g \in G \mid \{v, v^g\} \in E(X)\}$. Then:

Lemma 2.1:

(a) *D is a union of double cosets HaH of H in G*

(b) *D is closed under taking inverses*

(c) *v^x is adjacent to v^y in X if and only if $xy^{-1} \in D$*

(d) *The valency of X is the number of right cosets Hg contained in D*

(e) *X is connected if and only if D generates G.*

PROOF.

(a) If $a \in D$ then for all $h, h' \in H$ we have $v^{hah'} = v^{ah'} = (v^a)^{h'}$, which is the image of a neighbour of v under an automorphism fixing v, and hence a neighbour of v, so $hah' \in D$. Thus $HaH \subseteq D$ whenever $a \in D$, and so D is the union of all such double cosets of H.

(b) If $a \in D$ then $\{v^a, v\} \in E(X)$, and hence $\{v, v^{a^{-1}}\} = \{v^a, v\}^{a^{-1}} \in E(X)$, so $a^{-1} \in D$.

(c) $\{v^y, v^x\} \in E(X) \iff \{v, v^{xy^{-1}}\}^y \in E(X) \iff \{v, v^{xy^{-1}}\} \in E(X) \iff xy^{-1} \in D$.

(d) By vertex-transitivity, the valency of X is the number of neighbours of v. These neighbours are all of the form v^a for $a \in D$, and if $v^a = v^{a'}$ for $a, a' \in D$ then $v^{a'a^{-1}} = v$ so $a'a^{-1} \in G_V = H$, or equivalently, $a' \in Ha$, and conversely, if $a' = ha$ where $h \in H$, then $v^{a'} = v^{ha} = v^a$. Hence this valency equals the number of right cosets of H contained in D.

(e) Neighbours of v are of the form v^a where $a \in D$, and their neighbours are of the form $v^{a'a}$ where $a, a' \in D$. By induction, vertices at distance at most k from v are of the form $v^{a_k a_{k-1} \dots a_2 a_1}$ where $a_i \in D$ for $1 \le i \le k$. It follows that X is connected if and only if every vertex can be written in this form (for some k), or equivalently, if and only if every element of G can be written as a product of elements of D. □

Lemma 2.2: *X is arc-transitive if and only if the stabiliser H of a vertex v of X is transitive on the neighbours of v.*

PROOF. If X is arc-transitive, then for any two neighbours w and w' of v, there exists an automorphism $g \in G$ taking (v, w) to (v, w'). Any such g lies in G_v, and takes w to w', and it follows that G_v is transitive on the set $X(v)$ of all neighbours of v. Conversely, suppose that $H = G_v$ is transitive on $X(v)$. Then for any arcs (v, w) and (v', w'), some $g \in G$ takes v to v', and if the pre-image of w' under g is w'', then also some $h \in G_v$ takes w to w''. From these it follows that $(v, w)^{hg} = (v, w'')^g = (v', w')$. Thus X is arc-transitive. □

Lemma 2.3: *X is arc-transitive if and only if $D = HaH$ for some $a \in G \setminus H$, indeed if and only if $D = HaH$ for some $a \in G$ such that $a \notin H$ but $a^2 \in H$.*

PROOF. By Lemma 2.1, we know that X is arc-transitive if and only if $H = G_v$ is transitive on the neighbours of v, which occurs if and only if every neighbour of v is of the form w^h for some $w \in X(v)$ and some $h \in G_v = H$. By taking $v^a = w$, we find the equivalent condition that $D = HaH$ for some $a \in G \setminus H$.

For the second part, note that $a^{-1} \in D = HaH$, so $a^{-1} = hah'$ for some $h, h' \in H$. But then $aha = (h')^{-1}$, so $(ah)^2 = (h')^{-1}h \in H$, and also $H(ah)H = HahH = HaH = D$, so we can replace a by ah and then assume that $a^2 \in H$ (and still $a \notin H$). $\square$

Constructions: The observations in the preceding lemmas can be turned around to produce *constructions* for vertex-transitive and arc-transitive graphs, as follows.

Let G be any group, H any subgroup of G, and D any union of double cosets of G such that $H \cap D = \emptyset$, and D is closed under taking inverses. [Note: there is also a construction for vertex-transitive digraphs that does not assume D is inverse-closed.]

Now **define** a graph $X = X(G,H,D)$ by taking $V = V(X)$ to be the right coset space $(G:H) = \{Hg : g \in G\}$, and $E = E(X)$ to be the set of all pairs of the form $\{Hx, Hax\}$ where $a \in D$ and $x \in G$. [This construction is due to Sabidussi (1964)]

The adjacency relation is symmetric, since $Hx = Ha(a^{-1}x)$, and so this is an undirected simple graph. Also right multiplication gives an action of G on X, with $g \in G$ taking a vertex Hx to the vertex Hxg, and an edge $\{Hx, Hax\}$ to the edge $\{Hxg, Haxg\}$. This action is transitive on vertices, since g takes H to Hg for any $g \in G$. The stabiliser of the vertex H is $\{g \in G \mid Hg = H\}$, which is the subgroup H itself (since $Hg = H$ if and only if $g \in H$). Valency and connectedness are as in Lemma 2.1.

Note, however, that the action of G on $X(G,H,D)$ need not be faithful: the kernel K of this action is the *core* of H (the intersection of all conjugates $g^{-1}Hg$ of H) in G. Similarly, the group G/K induced on $X(G,H,D)$ need not be the full automorphism group $\mathrm{Aut}\,X$; it is often possible that the graph admits additional automorphisms.

Cayley graphs: Given a group G and a set D of elements of G, the *Cayley graph* $\mathrm{Cay}(G,D)$ is the graph with vertex-set G, and edge set $\{\{x, ax\} : x \in G, a \in D\}$. Note that this is a special case of the above, with $H = \{1\}$.

In particular, $\mathrm{Cay}(G,D)$ is vertex-transitive, and the group G acts faithfully and regularly on the vertex-set, but is not necessarily the full automorphism group. For example, a *circulant* (which is a Cayley graph for a cyclic group) can often have more than just simple rotations. Similarly, the n-dimensional hypercube Q_n is the Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^n, B)$ where B is the standard basis (of elementary vectors) for $\mathbb{Z}_2^n$, but $\mathrm{Aut}\,Q_n \cong \mathbb{Z}_2 \wr S_n \cong \mathbb{Z}_2^n \rtimes S_n$.

1.3 s-arc-transitivity (and Theorems of Tutte and Weiss)

As defined earlier, an *s-arc* in a graph X is a sequence $(v_0, v_1, \dots, v_s)$ of $s+1$ vertices of X in which any 2 consecutive vertices are adjacent and any 3 consecutive vertices are distinct. The graph X is called *s-arc-transitive* if $\mathrm{Aut}\,X$ is transitive on the s-arcs in X.

Lemma 3.1: *Let X be a vertex-transitive graph of valency $k > 2$, and let $G = \mathrm{Aut}\,X$. Then X is 2-arc-transitive if and only if the stabiliser G_v of a vertex v is 2-transitive on the k neighbours of v.*

Proof. If X is 2-arc-transitive, then for any two ordered pairs (u_1, w_1) and (u_2, w_2) of neighbours of v, some automorphism $g \in G$ takes the 2-arc (u_1, v, w_1) to the 2-arc

$$(u_2, v, w_2),$$

in which case g fixes v and g takes (u_1, w_1) to (u_2, w_2); hence G_v is 2-transitive on the neighbourhood $X(v)$. Conversely, suppose G_v is 2-transitive on $X(v)$, and let (u, v, w) and (u', v', w') be any two 2-arcs in X. Then by vertex-transitivity, some $g \in G$ takes v to v', and then if g takes u' to u'' and w' to w'', say, then some $h \in G_v$ takes (u, v) to (u'', v''), in which case $(u, v, w)^{hg^{-1}} = (u'', v, w'')^{g^{-1}} = (u', v', w')$; hence X is 2-arc-transitive. $\square$

Exercise 11: For a vertex-transitive graph X of valency 3, what are the possibilities for the permutation group induced on $X(v)$ by the stabiliser G_v in $G = \mathrm{Aut}\,X$ of a vertex v? Which of these correspond to arc-transitive actions?

Exercise 12: For an arc-transitive graph X of valency 4, what are the possibilities for the permutation group induced on $X(v)$ by the stabiliser G_v in $G = \mathrm{Aut}\,X$ of a vertex v?

Lemma 3.2: *Let X be a vertex-transitive graph of valency $k > 2$, and let $G = \mathrm{Aut}\,X$. Then X is $(s+1)$-arc-transitive if and only if X is s-arc-transitive and the stabiliser G_σ of an s-arc $\sigma = (v_0, v_1, \ldots, v_s)$ is transitive on $X(v_s) \setminus \{v_{s-1}\}$ (the set of $k-1$ neighbours of v_s other than v_{s-1}).*

Proof. If X is $(s+1)$-arc-transitive, then for any s-arc $\sigma = (v_0, v_1, \ldots, v_s)$ and any vertices w and w' in $X(v_s) \setminus \{v_{s-1}\}$, some automorphism $g \in G$ takes the $(s+1)$-arc $(v_0, v_1, \ldots, v_s, w)$ to the $(s+1)$-arc $(v_0, v_1, \ldots, v_s, w')$, in which case g fixes σ, and g takes w to w'; hence G_σ is transitive on $X(v_s) \setminus \{v_{s-1}\}$. Conversely, suppose G_σ is transitive on $X(v_s) \setminus \{v_{s-1}\}$, and let $(v_0, v_1, \ldots, v_s, v_{s+1})$ and $(w_0, w_1, \ldots, w_s, w_{s+1})$ be any two $(s+1)$-arcs in X. Then by s-arc-transitivity, some $g \in G$ takes $(v_0, v_1, \ldots, v_s)$ to $(w_0, w_1, \ldots, w_s)$, and then if g takes w_{s+1} to w', say, then some $h \in G_\sigma$ takes v_{s+1} to w', in which case

$$(v_0, v_1, \ldots, v_s, v_{s+1})^{hg^{-1}} = (v_0, v_1, \ldots, v_s, w')^{g^{-1}} = (w_0, w_1, \ldots, w_s, w_{s+1});$$

hence X is $(s+1)$-arc-transitive. $\square$

The simple cycle C_n (which has valency 2) is s-arc-transitive for all $s \geq 0$, as is the union of more than one copy of C_n. This case is somewhat exceptional. For $k > 2$, there is an upper bound on values of s for which there exists a finite s-arc-transitive graph of valency k, as shown by the theorems of Tutte and Weiss below.

The first theorem, due to W.T. Tutte, is for valency 3, and will be proved in Section 4. On the other hand, the second theorem, due to Richard Weiss, is for arbitrary valency $k \geq 3$, but its proof is much more difficult, and is beyond the scope of this course.

Theorem 3.3 [Tutte, 1959]: *Let X be a finite connected arc-transitive graph of valency 3. Then X is s-arc-regular (and so $|\mathrm{Aut}\,X| = 3 \cdot 2^{s-1} \cdot |V(X)|$) for some $s \leq 5$. Hence in particular, there are no finite 6-arc-transitive cubic graphs.*

The upper bound on s in Tutte's theorem is sharp; in fact, it is attained by infinitely many graphs, although these graphs are somewhat rare. The smallest example is given below.

Tutte's 8-cage: This is the smallest 3-valent graph of girth 8, and has 30 vertices. It can be constructed in many different ways. One way is as follows:

In the symmetric group S_6, there are $\binom{6}{2} = 15$ transpositions (2-cycles), and $5 \cdot 3 \cdot 1 = 15$ triple transpositions (sometimes called *synthemes*). Define a graph T by taking these 30 permutations as the vertices, and joining each triple transposition $(a,b)(c,d)(e,f)$ by an edge to each of its three transpositions (a,b), (c,d) and (d,e).

The resulting graph T is *Tutte's 8-cage.* It is 3-valent, bipartite and connected, and the group S_6 induces a group of automorphisms of T by conjugation of the elements.

Exercise 13: Write down the form of a typical 5-arc $(v_0, v_1, \ldots, v_5)$ in Tutte's cage T with initial vertex v_0 being a transposition (a,b). Use this to prove that (a) the group S_6 is transitive on all such 5-arcs, and (b) T is not 6-arc-transitive.

Exercise 14: Prove that the girth (the length of the smallest cycle) of Tutte's 8-cage is 8.

Now the group S_6 is somewhat special among symmetric groups in that $\mathrm{Aut}\, S_6$ is twice as large as S_6. In fact, S_6 admits an *outer automorphism* that interchanges the 15 transpositions with the 15 triple transpositions, and interchanges the 40 3-cycles with the 40 double 3-cycles $(a,b,c)(d,e,f)$. Any such outer automorphism reverses a 5-arc in Tutte's 8-cage, and it follows that Tutte's 8-cage is 5-arc-transitive.

Note that Tutte's theorem actually puts a bound on the order of the stabiliser of a vertex (in the automorphism group of a finite symmetric 3-valent graph). The same thing does not hold for 4-valent symmetric graphs, as shown by the following.

Necklace/wreath graphs: Take a simple cycle C_n, where $n \geq 3$, with vertices labelled $0, 1, 2, \ldots, n-1$ in cyclic order, and then replace every vertex j by a pair of vertices u_j and v_j, and join every such u_j and every such v_j by an edge to each of the four vertices u_{j-1}, v_{j-1}, u_{j+1} and v_{j+1}, with addition and subtraction of subscripts taken modulo n. The resulting 4-valent graph (called a 'necklace' or 'wreath' graph) has $2n$ vertices, and is arc-transitive, with automorphism group isomorphic to the wreath product $S_2 \wr D_n \cong (S_2)^n \rtimes D_n$. In particular, the stabiliser of any vertex has order 2^n, which is unbounded.

Exercise 15: What is the largest value of s for which the above graph (on $2n$ vertices) is s-arc-transitive?

It is also worth noting here that vertex-stabilisers are bounded for the automorphism groups of maps. A *map* is an embedding of a connected graph or multigraph on a surface, dividing the surface into simply-connected regions, called the *faces* of the map. By definition, an automorphism of a map M preserves incidence between vertices, edges and faces of M, and it follows that if a vertex v has degree k, then the stabiliser of v in $\mathrm{Aut}\, M$ is a subgroup of the dihedral group D_k. The most highly symmetric maps are called *rotary*, or *regular*.

Theorem 3.4 [Weiss, 1981]: *Let X be a finite connected s-arc-transitive graph of valency $k \geq 3$. Then $s \leq 7$, and if $s = 7$ then $k = 3^t + 1$ for some t. Hence in particular, there are no finite 8-arc-transitive graphs of valency k whenever $k > 2$.*

As with Tutte's theorem, the upper bound on s in Weiss's theorem is sharp. In fact, for every $t > 0$, the incidence graph of a generalised hexagon over $\mathrm{GF}(3^t)$ is a 7-arc-transitive graph of valency $3^t + 1$.

The proof of Weiss's theorem uses the fact that if X is s-arc-transitive for some $s \geq 2$, then X is 2-arc-transitive (by Lemma 3.2), and so the stabiliser in $G = \mathrm{Aut}\, X$ of a vertex

v of X is 2-transitive on the neighbourhood $X(v)$ of v (by Lemma 3.1). It then uses the classification of finite 2-transitive groups, obtained by Peter Cameron in 1981 using the classification of the finite simple groups (CFSG).

Finally in this section, we give something that is useful in proving Tutte's theorem (and in other contexts as well):

Lemma 3.5 (The 'even distance' lemma): *For any connected arc-transitive graph X, let $G^+ = \langle G_v, G_w \rangle$ be the subgroup of $G = \mathrm{Aut}\, X$ generated by the stabilisers G_v and G_w of any two adjacent vertices v and w. Then*

(a) *the orbit of v under G^+ contains all vertices at even distance from v,*

(b) *G^+ contains the stabiliser of every vertex of X,*

(c) *G^+ has index 1 or 2 in $G = \mathrm{Aut}\, X$, and*

(d) *$|G : G^+| = 2$ if and only if X is bipartite.*

PROOF. Let Ω and $\mho$ be the orbits of v and w under G^+. Then $\mho$ contains w^{G_v}, so contains all neighbours of v. Similarly, Ω contains all neighbours of w, so contains all vertices at distance 2 from v. Also G^+ contains their stabilisers; for example, if $h \in G^+$ takes v to z, then G^+ contains $h^{-1}G_v h = G_z$. Parts (a) and (b) now follow from these observations, by induction. By the same token, the orbit $\mho = w^{G^+}$ contains all vertices at even distance from w. Hence in particular, every vertex of X lies in $\Omega \cup \mho$. Also by the orbit-stabiliser theorem, $|G_v||\Omega| = |G^+| = |G_w||\mho|$, and then since $|G_v| = |G_w|$, this implies $|\Omega| = |\mho|$, and it follows that $|\Omega| = |\mho| = |V(X)|$ or $|V(X)|/2$. In the latter case, Ω and $\mho$ are disjoint, which happens if and only if X is bipartite (with parts Ω and $\mho$), and then also $|G| = |G_v||V(X)| = 2|G_v||\Omega| = 2|G^+|$, so G^+ has index 2 in G. On the other hand, in the former case, $\Omega = \mho = V(X)$ and $|G| = |G_v||V(X)| = |G_v||\Omega| = |G^+|$, and then $G^+ = G$. This proves parts (c) and (d). $\square$

1.4 Proof of Tutte's Theorem on Symmetric Cubic Graphs

Theorem [Tutte, 1959]: *Let X be a finite connected arc-transitive graph of valency 3. Then X is s-arc-regular (and so $|\mathrm{Aut}\, X| = 3 \cdot 2^{s-1} \cdot |V(X)|$) for some $s \le 5$. Hence in particular, there are no finite 6-arc-transitive cubic graphs.*

We will prove this in several stages, using only elementary theory of groups and graphs.

First, we let s be the largest positive integer t for which the graph X is t-arc-transitive, and let $G = \mathrm{Aut}\, X$.

Then we let $\sigma = (v_0, v_1, \ldots, v_s)$ be any s-arc in X, and consider the stabilisers in G of the 0-arc (v_0), the 1-arc (v_0, v_1), the 2-arc (v_0, v_1, v_2), and so on.

We use properties of these to show that X is s-arc-regular, and then by considering the smallest k for which the stabiliser in G of the k-arc $(v_0, v_1, \ldots, v_k)$ is abelian, we prove that $s \le 5$.

Lemma 4.1: *X is s-arc-regular.*

PROOF. We have assumed X is s-arc-transitive, so all we have to do is show that the stabiliser of an s-arc is trivial. So assume the contrary. Then every s-arc σ is preserved by some non-trivial automorphism f, and by conjugating by a 'shunt' if necessary, we can choose $\sigma = (v_0, v_1, \ldots, v_s)$ such that f moves one of the neighbours of v_s, say w. Then since f fixes v_s and its neighbour v_{s-1}, it must interchange w with the third neighbour w' of v_s. It follows that the stabiliser of the s-arc $\sigma = (v_0, v_1, \ldots, v_s)$ is transitive on the set of two $(s+1)$-arcs extending σ, namely $(v_0, v_1, \ldots, v_s, w)$ and $(v_0, v_1, \ldots, v_s, w')$. Hence X is $(s+1)$-arc-transitive, contradiction. □

Stabilisers

Let $\sigma = (v_0, v_1, \ldots, v_s)$ be any s-arc of X, and let $G = \mathrm{Aut}\, X$, and now define

$$\begin{array}{ccccc} H_s & = & G^{(0)} & = & G_{(v_0)} \\ H_{s-1} & = & G^{(1)} & = & G_{(v_0,v_1)} \\ \vdots & & \vdots & & \vdots \\ H_{s-k} & = & G^{(k)} & = & G_{(v_0,v_1,\ldots,v_k)} \\ \vdots & & \vdots & & \vdots \\ H_0 & = & G^{(s)} & = & G_{(v_0,v_1,\ldots,v_{s-1},v_s)} = \{1\}. \end{array}$$

Then working backwards, we find that $|H_j| = |G^{(s-j)}| = 2^j$ for $0 \le j < s$, while also $|H_s| = |G^{(0)}| = 3 \cdot 2^{s-1}$ and $|G| = 3 \cdot 2^{s-1} \cdot |V(X)|$.

Particular automorphisms

As before, let w and w' be the other two neighbours of v_s. Also let h and h' be the two automorphisms that take $\sigma = (v_0, v_1, \ldots, v_{s-1}, v_s)$ to $(v_1, v_2, \ldots, v_s, w)$ and $(v_1, v_2, \ldots, v_s, w')$ respectively, and define $x_0 = h'h^{-1}$ and $x_j = h^j x_0 h^{-j}$ for $1 \le j \le s$. Note that $x_0 = h'h^{-1}$ preserves $(v_0, v_1, \ldots, v_{s-1})$ and is non-trivial, so x_0 must swap v_s with the third neighbour of v_{s-1}; hence x_0 has order 2. It follows that every x_j has order 2.

Moreover, x_{j-1} preserves $(v_0, v_1, \ldots, v_{s-j})$ and swaps v_{s-j+1} with the third neighbour of v_{s-j}, for $1 \le j < s$. Hence in particular, $x_{j-1} \in H_j \setminus H_{j-1}$. Then since $|H_j| = 2|H_{j-1}|$, we find that H_j is generated by $\{x_0, x_1, \ldots, x_{j-1}\}$ for $1 \le j < s$. Similarly, x_{s-1} fixes v_0 but moves v_1, so $x_{s-1} \in H_s \setminus H_{s-1}$. Then since $|H_s : H_{s-1}| = 3$ (a prime), H_{s-1} is a maximal subgroup of H_s, so H_s is generated by $\{x_0, x_1, \ldots, x_{s-1}\}$.

Next, consider the subgroup G^* generated by $\{x_0, x_1, \ldots, x_s\}$. This contains $H_s = G_{v_0}$ and $\langle x_1, .., x_{s-1}, x_s \rangle = G_u$ where $u^h = v_0$, and so by the 'even distance' lemma, G^* is a subgroup of index 1 or 2 in G (the one we called G^+ earlier). Hence in particular, $|G^*| = 3 \cdot 2^{s-1} \cdot |V(X)|$ or half of that. Finally, since h moves v_0 to v_1 (which is at distance 1 from v_0), we find that $G = \langle h, G^* \rangle = \langle h, x_0, x_1, \ldots, x_s \rangle = \langle h, x_0 \rangle$.

We can summarise this in the following lemma.

Lemma 4.2:

(a) $H_j = \langle x_0, x_1, \ldots, x_{j-1} \rangle$ *for* $1 \le j \le s$,

(b) $G^+ = \langle x_0, x_1, \ldots, x_s \rangle$, *and*

(c) $G = \langle h, x_0, x_1, \ldots, x_s \rangle = \langle h, x_0 \rangle$.

Note that $H_1 = \langle x_0 \rangle$ and $H_2 = \langle x_0, x_1 \rangle$ are abelian, with orders 2 and 4 respectively.

Define λ to be the largest value of j for which H_j is abelian. We will show that $\frac{2}{3}(s-1) \le \lambda < \frac{1}{2}(s+2)$ whenever $s \ge 4$, and hence that $s \le 5$ or $s = 7$, and then we will eliminate the possibility that $s = 7$.

Lemma 4.3: *If $s \ge 4$, then $2 \le \lambda < \frac{1}{2}(s+2)$.*

PROOF. Assume the contrary. We know that $\lambda \ge 2$, so the assumption implies $2\lambda \ge s+2$, and hence $\lambda - 1 \ge s - \lambda + 1$. Now $H_\lambda = \langle x_0, x_1, \dots, x_{\lambda-1}\rangle$ is abelian, and therefore so is its conjugate $h^{s-\lambda+1} H_\lambda h^{-(s-\lambda+1)} = \langle x_{s-\lambda+1}, x_{s-\lambda+2}, \dots, x_s\rangle$. Then since $\lambda-1 \ge s-\lambda+1$, both of these contain $x_{\lambda-1}$, and also together they generate $\langle x_0, x_1, \dots, x_s\rangle = G^+$. It follows that $x_{\lambda-1}$ commutes with every element of G^+. In particular, $x_{\lambda-1}$ commutes with h^2 (which lies in G^+ since $|G : G^+| \le 2$). But that implies $x_{\lambda-1} = h^2 x_{\lambda-1} h^{-2} = x_{\lambda+1}$, and then conjugating by $h^{\lambda-1}$ gives $x_0 = x_2$, contradiction. □

Lemma 4.4: *The centre of $H_j = \langle x_0, x_1, \dots, x_{j-1}\rangle$ is generated by $\{x_{j-\lambda}, \dots, x_{\lambda-1}\}$, for $\lambda \le j < 2\lambda$.*

PROOF. Every element x of H_j can be written uniquely in the form $x = x_{i_1} x_{i_2} \dots x_{i_r}$ with $0 \le i_1 < i_2 < \dots < i_r \le j-1$. Now $[x_{i_1}, x_{i_1+\lambda}] \ne 1$ since otherwise $[x_0, x_\lambda] = 1$ and then x_λ commutes with $x_0, x_1, \dots, x_{\lambda-1}$, so $H_{\lambda+1} = \langle x_0, x_1, \dots, x_\lambda\rangle$ is abelian, contradiction. Thus $x_{i_1+\lambda} \notin Z(H_j)$ when $i_1+\lambda < j$. Similarly $[x_{i_r}, x_{i_r-\lambda}] \ne 1$ when $i_r - \lambda \ge 0$. Hence if $x \in Z(H_j)$ then $x = x_{i_1} x_{i_2} \dots x_{i_r}$ where $i_1 \ge j-\lambda$ and $i_r < \lambda$.

Conversely, if $0 \le j-\lambda \le i < \lambda \le j$ then x_i commutes with all of $x_0, x_1, \dots, x_{\lambda-1}$, because $H_\lambda = \langle x_0, x_1, \dots, x_{\lambda-1}\rangle$ is abelian, and with all of $x_\lambda, \dots, x_{j-1}$, since $h^\lambda H_\lambda h^{-\lambda} = \langle x_\lambda, \dots, x_{2\lambda-1}\rangle$ is abelian (and $\lambda \le j < 2\lambda$). Thus every such element $x_{i_1} x_{i_2} \dots x_{i_r}$ is central in H_j. □

Lemma 4.5: *The derived subgroup of $H_{j+1} = \langle x_0, x_1, \dots, x_j\rangle$ is a subgroup of $\langle x_1, \dots, x_{j-1}\rangle$, for $1 \le j \le s-2$.*

PROOF. Each of $\langle x_1, \dots, x_j\rangle$ and $\langle x_0, \dots, x_{j-1}\rangle$ has index 2 in $H_{j+1} = \langle x_0, x_1, \dots, x_j\rangle$, and is therefore normal in H_{j+1}. Their intersection $\langle x_1, \dots, x_{j-1}\rangle$ is a normal subgroup, of index 4, and (so) the quotient is abelian. Thus $\langle x_1, \dots, x_{j-1}\rangle$ contains all commutators of elements of H_{j+1}, and hence contains the derived subgroup of H_{j+1}. □

Next, consider the element $[x_0, x_\lambda] = x_0^{-1} x_\lambda^{-1} x_0 x_\lambda = (x_0 x_\lambda)^2$. By Lemma 4.5, this lies in $\langle x_1, \dots, x_{\lambda-1}\rangle$, so can be written in the form $x_{i_1} \dots x_{i_r}$ with $1 \le i_1 < \dots < i_r \le \lambda - 1$.

We will take $\mu = i_1$ and $\nu = i_r$, and show that $\mu + \lambda \ge s-1$ and $2\lambda - \nu \ge s-1$, and hence that $\frac{2}{3}(s-1) \le \lambda$.

Lemma 4.6: *If $[x_0, x_\lambda]$ is written as $x_{i_1} \dots x_{i_r}$ with $0 < i_1 < \dots < i_r < \lambda$, then*
(a) $i_1 + \lambda \ge s-1$, *and* (b) $2\lambda - i_r \ge s-1$.

PROOF. Take $\mu = i_1$ and $\nu = i_r$, so that $0 < \mu \le \nu < \lambda$.

For (a), suppose that $\mu + \lambda \le s-2$. Then Lemma 4.5 implies that $[x_0, x_{\mu+\lambda}]$ lies in

$$\langle x_1, \dots, x_{\mu+\lambda-1}\rangle,$$

the centre of which is $\langle x_\mu, \dots, x_\lambda\rangle$. The latter contains x_λ and $x_\mu \dots x_\nu = [x_0, x_\lambda]$, so both of these commute with $[x_0, x_{\mu+\lambda}]$. This gives

$$\begin{aligned}
&[x_0,x_\lambda]^{x_{\mu+\lambda}} \;=\; [x_0^{\;x_{\mu+\lambda}},x_\lambda^{\;x_{\mu+\lambda}}] \;=\; [x_0^{\;x_{\mu+\lambda}},x_\lambda] \\
&=\; [x_0[x_0,x_{\mu+\lambda}],x_\lambda] \;=\; [x_0,x_{\mu+\lambda}]^{-1}x_0^{-1}x_\lambda^{-1}x_0[x_0,x_{\mu+\lambda}]x_\lambda \\
&=\; [x_0,x_{\mu+\lambda}]^{-1}[x_0,x_\lambda]x_\lambda^{-1}[x_0,x_{\mu+\lambda}]x_\lambda \\
&=\; [x_0,x_{\mu+\lambda}]^{-1}[x_0,x_\lambda][x_0,x_{\mu+\lambda}] \;=\; [x_0,x_\lambda],
\end{aligned}$$

and therefore $x_\mu \dots x_\nu = [x_0,x_\lambda]$ commutes with $x_{\mu+\lambda}$, contradiction.

Similarly, if $2\lambda-\nu \le s-2$ then $[x_0,x_{2\lambda-\nu}] \in \langle x_1,\dots,x_{2\lambda-\nu-1}\rangle$, the centre of which is $\langle x_{\lambda-\nu},\dots,x_\lambda\rangle$, so $[x_0,x_{2\lambda-\nu}]$ commutes with $x_{\lambda-\nu}$ and $x_{\mu+\lambda-\nu}\dots x_\lambda = h^{\lambda-\nu}(x_\mu\dots x_\nu)h^{\nu-\lambda} = h^{\lambda-\nu}[x_0,x_\lambda]h^{\nu-\lambda} = [x_{\lambda-\nu},x_{2\lambda-\nu}]$. This gives

$$\begin{aligned}
&[x_{\lambda-\nu},x_{2\lambda-\nu}]^{x_0} \;=\; [x_{\lambda-\nu}^{\;x_0},x_{2\lambda-\nu}^{\;x_0}] \;=\; [x_{\lambda-\nu},x_{2\lambda-\nu}^{\;x_0}] \\
&=\; [x_{\lambda-\nu},[x_0,x_{2\lambda-\nu}]x_{2\lambda-\nu}] \\
&=\; x_{\lambda-\nu}\,x_{2\lambda-\nu}[x_0,x_{2\lambda-\nu}]^{-1}x_{\lambda-\nu}[x_0,x_{2\lambda-\nu}]x_{2\lambda-\nu} \\
&=\; x_{\lambda-\nu}\,x_{2\lambda-\nu}\,x_{\lambda-\nu}\,x_{2\lambda-\nu} \;=\; [x_{\lambda-\nu},x_{2\lambda-\nu}],
\end{aligned}$$

so $x_{\mu+\lambda-\nu}\dots x_\lambda = [x_{\lambda-\nu},x_{2\lambda-\nu}]$ commutes with $x_{\mu+\lambda}$, contradiction. This proves part (b). □

Lemma 4.7: *If $s \ge 4$, then $\lambda \ge \frac{2}{3}(s-1)$.*

Proof. Lemma 4.6 gives $s-1-\lambda \le \mu \le \nu \le 2\lambda-s+1$, and then forgetting μ and ν and rearranging gives $2s-2 \le 3\lambda$. □

Lemma 4.8: *If $s \ge 4$, then $s = 4, 5$ or 7.*

Proof. By Lemmas 3 and 6 we have $\frac{2}{3}(s-1) \le \lambda < \frac{1}{2}(s+2)$. Forgetting λ and rearranging gives $4s-4 < 3s+6$, so $s < 10$, but on the other hand, for $s \in \{6,8,9\}$ there is no integer solution for λ, so $s = 4, 5$ or 7. □

Lemma 4.9: $s \ne 7$.

Proof. Assume that $s = 7$. Then $\lambda = 4$, and $2 \le \mu \le \nu \le 2$ so $\mu = \nu = 2$, which gives $[x_0,x_4] = x_2$. Next we consider $[x_0,x_5]$. By Lemma 4.5, this lies in $\langle x_1,x_2,x_3,x_4\rangle$.

Suppose that $(x_0x_5)^2 = [x_0,x_5]$ lies in $\langle x_1,x_2,x_3\rangle$. Then $x_5x_0x_5$ lies in $\langle x_0,x_1,x_2,x_3\rangle = H_4$ and so fixes vertex v_3 of our original 7-arc $\sigma = (v_0,v_1,\dots,v_7)$, and hence x_0 fixes $v_3^{\;x_5}$. Observe that x_5 fixes v_0 and v_1 but not v_2 or v_3, and so $v_2^{\;x_5}$ is the third neighbour of v_1, different from v_0 and v_2 but then also fixed by x_0. It follows that x_0 preserves the 7-arc $(v_3^{\;x_5},v_2^{\;x_5},v_1,v_2,v_3,v_4,v_5,v_6)$, contradiction.

Thus $[x_0,x_5] = yx_4$ for some $y \in \langle x_1,x_2,x_3\rangle$. In particular, y commutes with x_0 and x_4 (since $\lambda = 4$), and also $y^2 = 1$ since $\langle x_1,x_2,x_3\rangle$ is abelian. But now it follows that

$$\begin{aligned}
x_2 &= [x_0,x_4] = (x_0x_4)^2 = (x_0yx_4)^2 = (x_0[x_0,x_5])^2 \\
&= (x_5x_0x_5)^2 = x_5x_0^{\;2}x_5 = 1, \text{ a final contradiction.}
\end{aligned}$$

□

This completes the proof of Tutte's theorem.

1.5 Amalgams and Covers

Recall (from Section 3) that if X is an arc-transitive graph, with automorphism group G, and H is the stabiliser of a vertex v, then X may be viewed as a double coset graph $X(G,H,D)$ where $D = HaH$ for some $a \in G$ (moving v to a neighbour), with $a^2 \in H$.

Lemma 5.1: *In the arc-transitive graph $X(G,H,HaH)$, the stabiliser in G of the arc (H,Ha) is the intersection $H \cap a^{-1}Ha$, and the stabiliser of the edge $\{H,Ha\}$ is the subgroup generated by $H \cap a^{-1}Ha$ and a. In particular, the valency of $X(G,H,HaH)$ is equal to the index $|H : H \cap a^{-1}Ha|$.*

Proof. Let v be the vertex H. Then since $a^2 \in H$, the element a interchanges v with its neighbour $w = v^a$, and hence reverses the arc (v,w). Also $a^{-1}Ha$ is the stabiliser of the vertex $v^a = w$, so $H \cap a^{-1}Ha$ is the stabiliser of the arc (v,w). The rest follows easily from this (and transitivity of H on the neighbours of v). □

Amalgams for symmetric graphs

For the next part of this section, we will abuse notation and use V, E and A respectively for the stabilisers of the vertex v, the edge $\{v,w\}$, and the arc (v,w), where w is the neighbour of v that is interchanged with v by the arc-reversing automorphism a.

The triple (V,E,A) may be called an *amalgam*. Note that $V \cap E = A$. Note also that if X is connected, then $G = \langle HaH \rangle = \langle H, a \rangle = \langle H, H \cap a^{-1}Ha, a \rangle = \langle V, E \rangle$.

This amalgam *specifies the kind of group action on* X. For example, if X is 3-valent and $V \cong S_3 \cong D_3$ and $E \cong V_4$ (the Klein 4-group), with $A = V \cap E \cong C_2$, then the action of G on X is 2-arc-regular, with the element a being an involution (reversing the arc (v,w)).

Conversely, from any such triple (V,E,A) we can form the amalgamated free product $\mathcal{U} = V *_A E$ (of the groups V and E with their intersection $A = V \cap E$ as amalgamated subgroup), which is a kind of *universal* group for such actions.

Specifically, G is an arc-transitive group of automorphisms of the symmetric graph X, acting in the way that is specified, if and only if G is a quotient of $\mathcal{U}$ via some homomorphism which preserves the amalgam (that is, preserves the orders of V, E and A). When that happens, the homomorphism takes V, E and A (faithfully) to the stabilisers of some vertex v, incident edge $\{v,w\}$ and arc (v,w) respectively,

This gives a way of classifying such graphs, or finding all of the examples of small order.

Exercise 16: What is the amalgam for the action of $S_3 \wr S_2$ on the graph $K_{3,3}$? Is this the same as the amalgam for the Petersen graph?

In 1980, Djoković and Miller determined all possible amalgams for an arc-transitive action of a group on a 3-valent graph with finite vertex-stabiliser. There are precisely seven such amalgams, which they called $1'$, $2'$, $2''$, $3'$, $4'$, $4''$ and $5'$. In each case, the given number is the value of s for which the group acts regularly on s-arcs, and $'$ indicates that the group contains arc-reversing elements that are involutions (of order 2), while $''$ indicates that every arc-reversing element has order greater than 2. (Note that we require $a^2 \in H$ but not necessarily $a^2 = 1$.) The first examples of finite 3-valent graphs with full automorphism groups of the types $2''$ and $4''$ were found by Conder and Lorimer (1989).

The universal groups for the seven Djoković-Miller amalgams are now customarily denoted by G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2 and G_5, with s being the subscript, and with G_s and G_s^1 corresponding to s', and G_s^2 corresponding to s''.

For example, the group G_1 is the modular group $\langle\, h, a \mid h^3 = a^2 = 1\,\rangle$, which is the free product of $V = \langle h\rangle \cong C_3$ and $E = \langle a\rangle \cong C_2$ (with $A = V \cap E = \{1\}$ amalgamated). Quotients of this group (in which the orders of V and E are preserved) are groups that act regularly on the arcs of a connected 3-valent symmetric graph.

Similarly, G_2^1 is the extended modular group $\langle\, h, p, a \mid h^3 = p^2 = a^2 = (hp)^2 = (ap)^2 = 1\,\rangle$, which is the amalgamated free product of $V = \langle h, p\rangle \cong S_3$ and $E = \langle p, a\rangle \cong V_4$ with amalgamated subgroup $A = V \cap E = \langle p\rangle \cong C_2$, while on the other hand, G_2^2 is the group $\langle\, h, p, a \mid h^3 = p^2 = a^4 = (hp)^2 = a^2p = 1\,\rangle$, which is the amalgamated free product of $V = \langle h, p\rangle \cong S_3$ and $E = \langle a\rangle \cong C_4$ with amalgamated subgroup $A = V \cap E = \langle p\rangle = \langle a^2\rangle \cong C_2$.

The group G_5 has a presentation on five generators h, p, q, r, s and a, obtainable from the amalgam $5'$, with $V = \langle h, p, q, r, s\rangle \cong S_4 \times C_2$ (of order 48), and $E = \langle p, q, r, s, a\rangle \cong D_4 \rtimes V_4$, and $A = V \cap E = \langle p, q, r, s\rangle \cong D_4 \times C_2$ (of order 16). This was used by Conder (1988) to prove that for all but finitely many n, both A_n or S_n occur as the automorphism groups of 5-arc-transitive cubic graphs.

Relationships between these seven groups were investigated by Djoković and Miller (1980) and Conder and Lorimer (1989), and recently by Conder and Nedela (2009) to refine the Djoković-Miller classification of arc-transitive group actions on symmetric cubic graphs.

Exercise 17: Find an example of a symmetric cubic graph that admits actions of arc-transitive groups of types $1'$, $2'$, $2''$ and $3'$.

Exercise 18: Can you find an example of a symmetric cubic graph that admits an action by an arc-transitive group of type $3'$ but not one of type $1'$?

Also in 1987, Richard Weiss identified the amalgams for several different kinds of s-arc-transitive group actions on **graphs of valency greater than 3**.

For example, Weiss produced one that gives the universal group for 7-arc-transitive group actions on 4-valent graphs, with V a group of order 11664 (being an extension of a group of order 3^5 by GL(2,3)), and E a group of order 5832, with $A = V \cap E$ having index 4 in V and index 2 in E. This was used by Conder and Walker (1998) to prove the existence of infinitely many 7-arc-transitive 4-valent graphs (indeed with automorphism group A_n or S_n, for all but finitely many n).

Amalgams for semi-symmetric graphs

The same kind of thing happens for semi-symmetric graphs. These can be analysed in terms of amalgams (H, K, L) consisting of the stabilisers $H = G_v$ and $K = G_w$ of adjacent vertices and their intersection $L = G_v \cap G_w$, which is the stabiliser of the edge $\{v, w\}$, since semi-symmetric graphs are edge- but not arc-transitive.

For semi-symmetric 3-valent graphs, there are 15 different amalgams, determined by Goldschmidt (1980). These were used by Conder, Malnič, Marušič, Pisanski and Potočnik to find all semi-symmetric 3-valent graphs of small order (in 2001), and led to their discovery of the *Ljubljana graph*, which is a semi-symmetric 3-valent graph of order 112 with interesting properties.

Graph quotients and covers

If X and Y are graphs for which there exists a graph homomorphism from Y onto X, then X is called a *quotient* of Y, and if the homomorphism is locally bijective — that is, faithful on the neighbourhood of each vertex — then Y is called a *cover* of X.

Exercise 19: Show that K_4 is a quotient of the cube graph Q_3. [Hint: antipodes!]

There are various ways of constructing covers of a given connected graph X. Some involve *voltage graph* techniques, which can be roughly described as follows:

Choose a spanning tree for X, and a permutation group P on some set Ω, and assign elements of P to the co-tree edges (the edges not included in the spanning tree), with each such edge given a specific orientation, to make it an arc. Then take $|\Omega|$ identical copies of X, and for each co-tree arc (v,w), use the label π (from P) to define copies of the arc, from the vertex v in the jth copy of X to the vertex w in the (j^{π})th copy of X, for all $j \in \Omega$. This gives a cover of X, with *voltage group* P.

Exercise 20: Construct Q_3 as a cover of K_4, using $P = S_2$.

In the 1970s, John Conway used a covering technique to produce infinitely covers of Tutte's 8-cage, and hence prove (for the first time) that there are infinitely many finite 5-arc-transitive cubic graphs. [This is described in Bigg's book *Algebraic Graph Theory*.]

Another way to construct covers of a symmetric graph X is to use the universal group $\mathcal{U} = V *_A E$ associated with the action of some arc-transitive group G of automorphisms of X. The group G is a quotient $\mathcal{U}/K$ for some normal subgroup K of $\mathcal{U}$, and then *for any normal subgroup L of $\mathcal{U}$ contained in K, the quotient $\mathcal{U}/L$ is an arc-transitive group of automorphisms of some cover of X.*

1.6 Some Recent Developments

This final section describes a number of recent developments on topics mentioned earlier.

Foster census:

In the 1930s, Ronald M. Foster (a mathematician/engineer working for Bell Labs) began compiling a list of all known connected symmetric 3-valent graphs of order up to 512. This 'census' was published in 1988, and was remarkably good, with only a few gaps.

The Foster census was extended by Conder and Dobcsányi (2002), with the help of some computational group theory and distributed computing. The extended census filled the gaps in Foster's list, and took it further, up to order 768. This also produced the smallest symmetric cubic graph of Djoković-Miller type $2''$, on 448 vertices. (The previously smallest known example had order 6652800.) In 2011/12, with the help of a new algorithm for finding finite quotients of finitely-presented groups, Conder extended this census, to find all connected symmetric 3-valent graphs of order up to 10000.

Other such lists:

Primož Potočnik, Pablo Spiga and Gabriel Verret have developed new methods for finding all *vertex-transitive* cubic graphs of small orders, and in 2012 used these to find all such graphs of order up to 1280, as well as all symmetric 4-valent graphs of order up to 640 (using a relationship between these kinds of graphs).

In 2013, Conder and Potočnik extended the list of all *semi-symmetric* 3-valent graphs, up to order 10000 as well.

There are similar lists of arc transitive graphs embedded on surfaces as *regular maps*, with large automorphism group. See `www.math.auckland.ac.nz/~conder` for some of these.

Related open problems (concerning pathological examples):

(a) *What is the smallest symmetric cubic graph of Djoković-Miller type* $4''$*?*

Such a graph must be 4-arc-regular, but have no arc-reversing automorphisms of order 2. The smallest known example has order 5314410, with automorphism group an extension of $(C_3)^{11}$ by PGL(2, 9). There is another nice (but larger) example of order 20401920, with automorphism group the simple Mathieu group M_{24}.

(b) *What is the smallest* 5*-arc-transitive cubic graph* X *with the property that its automorphism group is the only arc-transitive group of automorphisms of* X*?*

The smallest known example has order 2497430038118400, with automorphism group $M_{24} \wr S_2$. Examples are known with an alternating group A_n as automorphism group, but the smallest such n is 26.

(c) *What is the smallest half-arc-transitive graph* X *for which the stabiliser of a vertex in* Aut X *is neither abelian nor dihedral?*

Conder and Potočnik found one recently of order $90 \cdot 3^{10}$, with vertex-stabiliser $D_4 \times C_2$.

Covers:

Cheryl Praeger and some of her colleagues have done a lot of work on decomposing and constructing symmetric graphs via their quotients, and are using this to form a (loose) classification of all 2-arc-transitive finite graphs.

Group-theoretic covering methods have been applied to find all symmetric (or semisymmetric) regular covers of various small graphs, with abelian covering groups. For example:

- Cyclic symmetric coverings of Q_3 [Feng & Wang (2003)],
- Cyclic symmetric coverings of $K_{3,3}$ [Feng & Kwak (2004)],
- Elementary abelian symmetric covers of the Petersen graph [Malnič & Potočnik (2006)],
- Semisymmetric elementary abelian covers of the Möbius-Kantor graph [Malnič, Marušič, Miklavič & Potočnik (2007)],
- Elementary abelian symmetric covers of the Pappus graph [Oh (2009)],
- Elementary abelian symmetric covers of the octahedral graph [Kwak & Oh (2009)],

- Elementary abelian symmetric covers of K_5 [Kuzman (2010)].

In joint work with PhD student Jicheng Ma (2009–2012) we now have all symmetric abelian regular covers of K_4, $K_{3,3}$, Q_3, the Petersen graph and the Heawood graph.

Degree-diameter problem:

The *degree-diameter problem* involves finding the largest (regular) connected graph with given vertex-degree d and diameter D; for example, the Petersen graph is the largest for $(d, D) = (3, 2)$. In his PhD thesis project (2005-2008), Eyal Loz used voltage graphs to find covers of various small vertex- and/or arc-transitive graphs that are now the best known graphs in over half of the cases in the degree-diameter table. For more information, see: `moorebound.indstate.edu/wiki/The_Degree_Diameter_Problem_for_General_Graphs`.

Locally arc-transitive graphs:

A semi-symmetric graph is not vertex-transitive, but nevertheless can have a high degree of symmetry (subject to that constraint). A graph X is *locally s-arc-transitive* if the stabiliser in $\mathrm{Aut}\,X$ of a vertex v is transitive on all s-arcs in X starting at v.

An unpublished theorem of Stellmacher (1996) states that *If X is a finite locally s-arc-transitive graph, then $s \leq 9$.* Until recently, the only known examples for $s = 9$ came from classical generalised octagons and their covers. Such graphs are semi-symmetric (and hence bipartite) but not regular: vertices in different parts can have different valencies.

The smallest example for $s = 9$ has order 4680, with vertices of valency 3 in one part and 5 in the other. Its automorphism group is ${}^2F_4(2)$ (a Ree simple group), with vertex-stabilisers H and K of orders 12288 and 20480, and arc/edge-stabiliser $L = H \cap K$ of order 4096. In response to a comment by Michael Giudici at Rogla in 2011, Conder proved that the amalgamated free product $H *_L K$ has all but finitely many alternating groups A_n as quotients. Hence there exist *infinitely locally 9-arc-transitive bipartite graphs with vertices of valency 3 in one part and 5 in the other.*

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Chapter 2

Imprimitive Permutation Groups

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SUMMARY

With the Classification of the Finite Simple Groups and the O'Nan-Scott Theorem, much detailed information concerning primitive permutation groups has now been obtained. While primitive permutation groups are interesting in their own right, primitive permutation groups are actually quite rare, with a "typical" transitive permutation group being imprimitive. However, primitive permutation groups are the building blocks of imprimitive permutation groups, and so are the building blocks of all transitive groups. We will discuss techniques to analyze imprimitive permutation groups (sometimes using the recently obtained detailed knowledge of primitive permutation groups), with an emphasis on determining information about the automorphism group of a vertex-transitive digraph.

2.1 Introduction

The O'Nan-Scott Theorem together with the Classification of the Finite Simple Groups is a powerful tool that give the structure of all primitive permutation groups, as well as their actions. This has allowed for the solution to many classical problems, and has opened the door to a deeper understanding of imprimitive permutation groups, as primitive permutation groups are the building blocks of imprimitive permutation groups. We first give a more or less standard introduction to imprimitive groups, and then move to less well-known techniques, with an emphasis on studying automorphism groups of graphs.

A few words about these lecture notes. The lecture notes are an “expanded" version of the lecture - some of the lecture will be basically exactly these lecture notes, but in many cases the proofs of some background results (typically those that in my view are those whose proofs are primarily checking certain computations) are given in these lecture notes but will not be given in the lectures due to time constraints. Also, the material is organized into sections by topic, not by lecture.

2.2 Basic Results on Imprimitive Groups

Definition 2.2.1 Let G be a transitive group acting on X. A subset $B \subseteq X$ is a **block** of G if whenever $g \in G$, then $g(B) \cap B = \emptyset$ or B. If $B = \{x\}$ for some $x \in X$ or $B = X$, then B is a **trivial block**. Any other block is nontrivial. If G has a nontrivial block then it is **imprimitive**. If G is not imprimitive, we say that G is **primitive**. Note that if B is a block of G, then $g(B)$ is also a block of B for every $g \in G$, and is called a **conjugate block of** B. The set of all blocks conjugate to B, denoted $\mathcal{B}$, is a partition of X, and $\mathcal{B}$ is called a **complete block system of** G.

There does not seem to be a standard term for what is called here a complete block system of G. Other authors use a **system of imprimitivity** or a G-**invariant partition** for this term.

Theorem 2.2.2 *Let $\mathcal{B}$ be a complete block system of G. Then every block in $\mathcal{B}$ has the same cardinality, say k. Further, if m is the number of blocks in $\mathcal{B}$ then mk is the degree of G.*

Theorem 2.2.3 *Let G be a transitive group acting on X. If $N \triangleleft G$, then the orbits of N form a complete block system of G.*

Proof. Let $x \in X$ and B the orbit of N that contains x, so that $B = \{h(x) : h \in N\}$. Let $g \in G$, and for $h \in N$, denote by h' the element of N such that $gh = h'g$. Note h' always exists as $N \triangleleft G$, and that $\{h' : h \in N\} = N$ as conjugation by g induces an automorphism of N. Then $g(B) = \{gh(x) : h \in N\} = \{h'g(x) : h \in N\} = \{h(g(x)) : h \in N\}$. Hence $g(B)$ is the orbit of N that contains $g(x)$, and as the orbits of N form a partition of X, $g(B) \cap B = \emptyset$ or B. Thus B is a block, and as every conjugate block $g(B)$ of B is an orbit of N, the orbits of N do indeed form a complete block system of G. □

Example 2.2.4 Define $\rho,\tau:\mathbb{Z}_2\times\mathbb{Z}_5\mapsto\mathbb{Z}_2\times\mathbb{Z}_5$ by $\rho(i,j)=(i,j+1)$ and $\tau(i,j)=(i+1,2j)$. Note that in these formulas, arithmetic is performed modulo 2 in the first coordinate and modulo 5 in the second coordinate. It is straightforward but tedious to check that $\langle\rho,\tau\rangle$ is a subgroup of the automorphism group of the Petersen graph with the labeling shown in Figure 2.1.

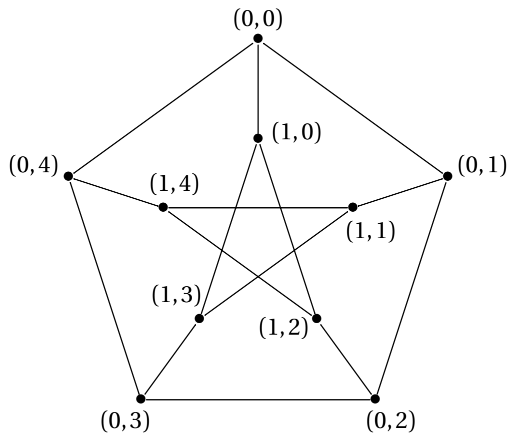

Figure 2.1: The Petersen graph.

Additionally, $\tau^{-1}(i,j)=(i-1,3j)$ as

$$\tau^{-1}\tau(i,j)=\tau^{-1}(i+1,2j)=(i+1-1,3(2j))=(i,j).$$

Also,

$$\tau^{-1}\rho\tau(i,j)=\tau^{-1}\rho(i+1,2j)=\tau^{-1}(i+1,2j+1)=(i+1-1,3(2j+1))=(i,j+3)=\rho^3(i,j)$$

and so $\langle\rho\rangle\triangleleft\langle\rho,\tau\rangle$. Then by Theorem 2.2.3 the orbits of $\langle\rho\rangle$, which are the sets $\{\{i,j\}:j\in\mathbb{Z}_5\}:i\in\mathbb{Z}_2\}$ form a complete block system of $\langle\rho,\tau\rangle$.

Although we will not show this here, the full automorphism group of the Petersen graph is primitive.

A complete block system of G formed by the orbits of normal subgroup of G is called a **normal complete block system of** G. Note that not every complete block system $\mathcal{B}$ of every transitive group G is a complete block system of G, as we shall see.

Now suppose that $G\le\mathcal{S}_n$ is a transitive group which admits a complete block system $\mathcal{B}$ consisting m blocks of size k. Then G has an **induced action on** $\mathcal{B}$, which we denote by $G/\mathcal{B}$. Namely, for specific $g\in G$, we define $g/\mathcal{B}(B)=B'$ if and only if $g(B)=B'$, and set $G/\mathcal{B}=\{g/\mathcal{B}:g\in G\}$. We also define the **fixer of** $\mathcal{B}$ **in** G, denoted $\mathrm{fix}_G(\mathcal{B})$, to be $\{g\in G:g/\mathcal{B}=1\}$. That is, $\mathrm{fix}_G(\mathcal{B})$ is the subgroup of G which fixes each block of $\mathcal{B}$ set-wise. Furthermore, $\mathrm{fix}_G(\mathcal{B})$ is the kernel of the induced homomorphism $G\to S_{\mathcal{B}}$, and as such is normal in G. Additionally, $|G|=|G/\mathcal{B}|\cdot|\mathrm{fix}_G(\mathcal{B})|$.

A transitive group G is **regular** if $\mathrm{Stab}_G(x)=1$ for any (and so all) x.

Theorem 2.2.5 *Let $G \leq \mathscr{S}_n$ be transitive with an abelian regular subgroup H. Then any complete block system of G is normal, and is formed by the orbits of a subgroup of H.*

PROOF. We only need show that $\mathrm{fix}_H(\mathscr{B})$ has orbits of size $|B|$, $B \in \mathscr{B}$. Now, $H/\mathscr{B}$ is transitive and abelian, and so $H/\mathscr{B}$ is regular (a transitive abelian group is regular as conjugation permutes the stabilizers of points - so in a transitive abelian group, point stabilizers are all equal). Then $H/\mathscr{B}$ has degree $|\mathscr{B}|$, and so there exists nontrivial $K \leq \mathrm{fix}_H(\mathscr{B})$ of order $|B|$. Then the orbits of K form a complete block system $\mathscr{C}$ of H by Theorem 2.2.3, and each block of $\mathscr{C}$ is contained in a block of $\mathscr{B}$. As K has order $|B|$, we conclude that $\mathscr{C} = \mathscr{B}$. □

Lemma 2.2.6 *Let G act transitively on X, and let $x \in X$. Let $H \leq G$ be such that* $\mathrm{Stab}_G(x) \leq H$. *Then the orbit of H that contains x is a block of G.*

PROOF. Set $B = \{h(x) : h \in H\}$ (so that B is the orbit of H that contains x), and let $g \in G$. We must show that B is a block of G, or equivalently, that $g(B) = B$ or $g(B) \cap B = \emptyset$. Clearly if $g \in H$, then $g(B) = B$ as B is the orbit of H that contains x and $x \in B$. If $g \notin H$, then towards a contradiction suppose that $g(B) \cap B \neq \emptyset$, with say $z \in g(B) \cap B$. Then there exists $y \in B$ such that $g(y) = z$ and $h, k \in H$ such that $h(x) = y$ and $k(x) = z$. Then

$$z = g(y) = gh(x) = k(x) = z,$$

and so $gh(x) = k(x)$. Thus $k^{-1}gh \in \mathrm{Stab}_G(x)$. This then implies that $g \in k \cdot \mathrm{Stab}_G(x) \cdot h^{-1} \leq H$, a contradiction. Thus if $g \notin H$, then $g(B) \cap B = \emptyset$, and B is a block of G. □

Example 2.2.7 Consider the subgroup of the automorphism group of the Petersen graph $\langle \rho\ \tau \rangle$ that we saw before. Straightforward computations will show that $|\tau| = 4$, and so $|\langle \rho, \tau \rangle| = 20$ as $|\rho| = 5$. By the Orbit-Stabilizer Theorem, we have that $\mathrm{Stab}_{\langle \rho, \tau \rangle}(0,0)$ has order 2, and as τ^2 stabilizes $(0,0)$, $\mathrm{Stab}_{\langle \rho, \tau \rangle}(0,0) = \langle \tau^2 \rangle$. Then $\langle \tau \rangle \leq \langle \rho, \tau \rangle$ and contains $\mathrm{Stab}_{\langle \rho, \tau \rangle}(0,0)$. Then the orbit of $\langle \tau \rangle$ that contains $(0,0)$ is a block of $\langle \rho, \tau \rangle$ as well. This orbit is $\{(0,0),(1,0)\}$. So the corresponding complete block system of $\langle \rho, \tau \rangle$ consists of the vertices of the "spoke" edges of the Petersen graph.

Just as we may examine the stabilizer of a point in a transitive group G, we may also examine the **stabilizer of the block** B in an imprimitive group G. It is denoted $\mathrm{Stab}_G(B)$, is a subgroup of G, and $\mathrm{Stab}_G(B) = \{g \in G : g(B) = B\}$.

Theorem 2.2.8 *Let G act transitively on X, and let $x \in X$. Let Ω be the set of all blocks B of G which contain x, and S be the set of all subgroups $H \leq G$ that contain* $\mathrm{Stab}_G(x)$. *Define $\phi : \Omega \to S$ by $\phi(B) = \mathrm{Stab}_G(B)$. Then ϕ is a bijection, and if $B, C \in \Omega$, then $B \subseteq C$ if and only if* $\mathrm{Stab}_G(B) \leq \mathrm{Stab}_G(C)$.

PROOF. First observe that $\mathrm{Stab}_G(x) \leq \mathrm{Stab}_G(B)$ for every block B with $x \in B$, so ϕ is indeed a map from Ω to S. We first show that ϕ is onto. Let $H \in S$ so that $\mathrm{Stab}_G(x) \leq H$. By Lemma 2.2.6, $B = \{h(x) : h \in H\}$ is a block of G. Then $H \leq \phi(B)$. Towards a contradiction, suppose there exists $g \in \phi(B)$ such that $g \notin H$. Then $g(B) = B$, and H is transitive in its action on B (Exercise 2.2.12). Hence there exists $h \in H$ such that $h(x) = g(x)$, and so

$h^{-1}g(x) = x \in \mathrm{Stab}_G(x) \leq H$. Thus $h^{-1}g \in H$ so $g \in H$, a contradiction. Thus $\phi(B) = H$ and ϕ is onto.

We now show that ϕ is one-to-one. Suppose $B, C \in \Omega$ and $\phi(B) = \phi(C)$. Then $\mathrm{Stab}_G(B) = \mathrm{Stab}_G(C)$. Towards a contradiction, suppose that $y \in B$ but $y \notin C$. As $\mathrm{Stab}_G(B)$ is transitive on B, there exists $h \in \mathrm{Stab}_G(B)$ such that $h(x) = y$. But then $h \in \mathrm{Stab}_G(C) = \mathrm{Stab}_G(B)$ and so y is in the orbit of $\mathrm{Stab}_G(C)$ that contains x, which is C, a contradiction. Thus ϕ is one-to-one and onto, and so a bijection.

Finally, it remains to show that if $B, C \in \Omega$, then $B \subseteq C$ if and only if $\mathrm{Stab}_G(B) \leq \mathrm{Stab}_G(C)$. First suppose that $\mathrm{Stab}_G(B) \leq \mathrm{Stab}_G(C)$. Then the orbit of $\mathrm{Stab}_G(C)$ that contains x certainly contains the orbit of $\mathrm{Stab}_G(B)$ that contains x, and so $B \subseteq C$. Conversely, suppose that $B \subseteq C$. Let $g \in \mathrm{Stab}_G(B)$. Then $g(x) \in B \subseteq C$, and so $x \in C \cap g(C)$. As C is a block of G, we have that $g(C) = C$ so that $g \in \mathrm{Stab}_G(C)$. Thus $\mathrm{Stab}_G(B) \leq \mathrm{Stab}_G(C)$. □

Theorem 2.2.9 *Let G be a transitive group acting on X. If $\equiv$ is an equivalence relation on X such that $x \equiv y$ if and only if $g(x) \equiv g(y)$ for all $g \in G$ (a G-**congruence**), then the equivalence classes of $\equiv$ form a complete block system of G.*

Proof. Let B_x be an equivalence class of $\equiv$ that contains x, and $x \in X$, $g \in G$. Then

$$\begin{aligned} g(B_x) &= \{g(y) : y \in X \text{ and } x \equiv y\} \\ &= \{g(y) : g(y) \equiv g(x)\} \\ &= B_{g(x)}. \end{aligned}$$

As the equivalence classes of $\equiv$ form a partition of X, it follows that $g(B_x) \cap B_x = \emptyset$ or B_x, and so B_x is a block of G. Also, as $g(B_x) = B_{g(x)}$, the set of all blocks conjugate to B_x is just the set of equivalence classes of $\equiv$. □

A common application of the above result is to stabilizers of points, as in a transitive group, any two point stabilizers are conjugate (Exercise ?.?).

Exercise 2.2.10 *Verify that if B is a block of G, then $g(B)$ is also a block of G for every $g \in G$.*

Exercise 2.2.11 *Verify that if $\mathcal{B}$ is a complete block system of G acting on X, then $\mathcal{B}$ is a partition of X.*

Exercise 2.2.12 *Let G act transitively on X, and suppose that B is a block of G. Then* $\mathrm{Stab}_G(B)$ *is transitive on B.*

Exercise 2.2.13 *Show that a transitive group of prime degree is primitive.*

Exercise 2.2.14 *Let $G \leq \mathcal{S}_n$ with $\mathcal{B}$ a complete block system of G. If $\phi \in \mathcal{S}_n$, then $\phi(\mathcal{B})$ is a $\phi G \phi^{-1}$-invariant partition.*

Exercise 2.2.15 *A group G acting on X is* **doubly-transitive** *if whenever $(x_1, y_1), (x_2, y_2) \in X \times X$ such that $x_1 \neq y_1$ and $x_2 \neq y_2$, then there exists $g \in G$ such that $g(x_1, y_1) = (x_2, y_2)$. Show that a doubly-transitive group is primitive.*

Exercise 2.2.16 *Let $G \leq \mathcal{S}_n$ contain a regular cyclic subgroup $R = \langle (0\ 1\ \ldots\ n-1)$ and admit a complete block system $\mathcal{B}$ consisting of m blocks of size k. Show that $\mathcal{B}$ consists of cosets of the unique subgroup of $\mathbb{Z}_n$ of order k.*

Exercise 2.2.17 *Let p and q be distinct primes such that q divides $p-1$. Determine the number of complete block systems of G_L where G is the nonabelian group of order pq that consist of blocks of cardinality q and of cardinality p.*

Exercise 2.2.18 *Let G be a transitive group of square-free degree (an integer that is* **square-free** *is not divisible by the square of any prime). Show that G has at most one normal G-invariant partition with blocks of prime size p. (Hint: Suppose there are at least two such G-invariant partitions $\mathcal{B}_1$ and $\mathcal{B}_2$. Consider what happens to* $\mathrm{fix}_G(\mathcal{B}_2)$ *in $G/\mathcal{B}_1$.)*

Exercise 2.2.19 *Let $G \leq \mathcal{S}_n$ be transitive. Show that G is primitive if and only if* $\mathrm{Stab}_G(x)$ *is a maximal subgroup of G for every $x \in \mathbb{Z}_n$.*

2.3 Notions of “Sameness" of Permutation Groups

First, a group G may be represented as a permutation group in different ways. We first need to be able to distinguish when two such representations are essentially the same, or are different.

Definition 2.3.1 We say that the action of G on sets A and B are **permutation equivalent** if there exists a bijection $\lambda : A \to B$ such that $\lambda(g(x)) = g(\lambda(x))$ for all $x \in A$ and $g \in G$.

On obvious way in which G can have equivalent actions on different sets A and B are if $|A| = |B|$, and we simply relabel the elements of A with elements of B. In this case, the defining equation $\lambda(g(x)) = g(\lambda(x))$ for all $x \in A$ and $g \in G$ states that if we apply g to x and then relabel, then this is the same as if we relabel x and then apply g to the relabelled x. For our purposes, we will be concerned with transitive groups. The following result is used in practice to determine if two actions of G are equivalent.

Theorem 2.3.2 *Let G act transitively on A and B. Then the action of G on A is equivalent to the action of G on B if and only if the stabilizer in G of a point in A is the stabilizer of a point in B.*

Proof. Suppose that the action of G on A is equivalent to the action of G on B. Then there exists a bijection $\lambda : A \to B$ such that $\lambda(g(x)) = g(\lambda(x))$ for all $x \in A$ and $g \in G$. Let $K = \mathrm{Stab}_G(z)$, where $z \in B$, and $y \in A$ such that $\lambda(y) = z$. Let $k \in K$. As $k(z) = z$, we have that

$$\lambda(k(y)) = k(\lambda(y)) = k(z) = z.$$

As λ is a bijection, $k(y) = \lambda^{-1}(z) = y$, and so k stabilizes y. Thus $K \leq \mathrm{Stab}_G(y)$, and as G is transitive on A and B and $|A| = |B|$, by the Orbit-Stabilizer Theorem we see that $K = \mathrm{Stab}_G(y)$.

Now suppose that $\mathrm{Stab}_G(a) = \mathrm{Stab}_G(b)$ for some $a \in A$ and $b \in B$. Define $\lambda : A \mapsto B$ by $\lambda(g(a)) = g(b)$. We first need to show that λ is well-defined. That is, that regardless of choice of g, $\lambda(x)=y$, $x\in A$, $y\in B$, is the same. So we need to show that if $g(a)=h(a)$, then $g(b)=\lambda(g(a))=\lambda(h(a))=h(b)$. Now,

$$\begin{aligned} g(a)=h(a) &\Rightarrow h^{-1}g(a)=a \\ &\Rightarrow h^{-1}g\in \mathrm{Stab}_G(a) \\ &\Rightarrow h^{-1}g(b)=b \\ &\Rightarrow g(b)=h(b) \\ &\Rightarrow \lambda(g(a))=\lambda(h(a)) \end{aligned}$$

and so λ is indeed well-defined. Also, as G is transitive on A, λ has domain A, and as G is transitive on B, λ is surjective, and hence bijective. Finally, let $x\in A$. Then there exists $h_x\in G$ such that $h_x(a)=x$. Then

$$\begin{aligned} \lambda(g(x)) &= \lambda(gh_x(a)) \\ &= gh_x(b) \\ &= g\lambda(h_x(a)) \\ &= g\lambda(x). \end{aligned}$$

□

Definition 2.3.3 Let $G\le S_A$ and $H\le S_B$. Then G and H are **permutation isomorphic** if there exists a bijection $\lambda: A\to B$ and a group isomorphism $\phi: G\to H$ such that $\lambda(g(x))=\phi(g)(\lambda(x))$ for all $x\in A$ and $g\in G$.

Intuitively, in addition to relabeling the set on which G acts (via λ), we also relabel the group (via the homomorphism ϕ).

Theorem 2.3.4 *Let G be a transitive group acting on A that admits a complete block system $\mathcal{B}$. Then the action of* $\mathrm{Stab}_G(B)$ *on B and the action of* $\mathrm{Stab}_G(B')$ *on B' are permutation isomorphic. Additionally, the action of* $\mathrm{fix}_G(\mathcal{B})$ *on B is permutation isomorphic to the action of* $\mathrm{fix}_G(\mathcal{B})$ *on B'.*

Proof. Let $\ell\in G$ such that $\ell(B)=B'$. Define $\lambda: B\to B'$ by $\lambda(x)=\ell(x)$. As ℓ maps B bijectively to B', λ is a bijection. Define $\phi: \mathrm{Stab}_G(B)\to \mathrm{Stab}_G(B')$ by $\phi(g)=\ell g\ell^{-1}$. As ϕ is obtained by conjugation, ϕ is a group isomorphism. Let $g\in \mathrm{Stab}_G(B)$, and $x\in B$. Then

$$\lambda(g(x))=\ell g(x)=\ell g\ell^{-1}\ell(x)=\phi(g)\lambda(x),$$

and so the action of $\mathrm{Stab}_G(B)$ on B is permutation isomorphic to the action of $\mathrm{Stab}_G(B')$ on B'. Analogous arguments will show that the action of $\mathrm{fix}_G(\mathcal{B})$ on B is permutation isomorphic to the action of $\mathrm{fix}_G(\mathcal{B})$ on B'. □

2.4 An Example of Inequivalent Actions: The Automorphism Group of the Heawood Graph

For a subspace S of $\mathbb{F}_q^n$, we denote by $S^\perp$ the **othogonal complement of S**. That is, $S^\perp = \{w \in \mathbb{F}_q^n : w \cdot v = 0 \text{ for every } v \in S\}$. Recall that $S^\perp$ is a subspace of the vector space $\mathbb{F}_q^n$. A **line** in $\mathbb{F}_q^n$ is a one-dimensional subspace, while a **hyperplane** is the orthogonal complement of a line (so a subspace of $\mathbb{F}_q^n$ of dimension $n-1$). Note that the number of lines and hyperplanes of $\mathbb{F}_q^n$ are the same. In the case of $\mathbb{F}_2^3$ which contains 8 elements, any nonzero vector gives rise to a line, so there are 7 lines and 7 hyperplanes.

Consider the graph whose vertex set is the lines and hyperplanes of $\mathbb{F}_2^3$, and a line is adjacent to a hyperplane if and only if the line is contained in the hyperplane. We obtain the following graph, which is isomorphic to the Heawood graph:

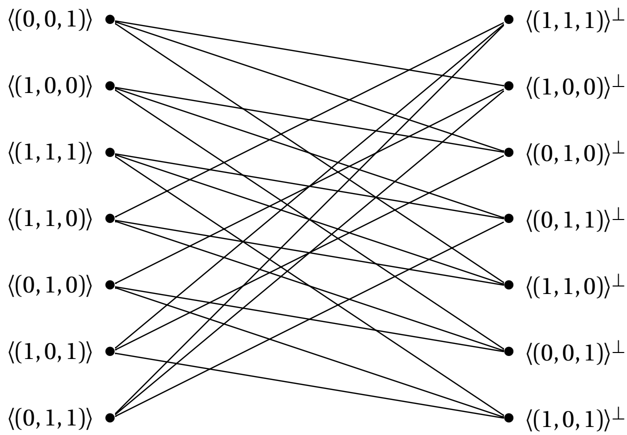

Figure 2.2: The Heawood graph labeled with the lines and hyperplanes of $\mathbb{F}_2^3$

Recall that $\mathrm{GL}(n,\mathbb{F}_q)$ is the **general linear group of dimension n over the field $\mathbb{F}_q$**. That is $\mathrm{GL}(n,\mathbb{F}_q)$ is the group of all invertible $n \times n$ matrices with entries in $\mathbb{F}_q$, with binary operation multiplication. In the literature, it is common to see $\mathrm{GL}(n,q)$ written in place of $\mathrm{GL}(n,\mathbb{F}_q)$, a convention that we will follow. Of course, a linear transformation maps lines to lines, so we can consider the action of $\mathrm{GL}(3,2)$ on the lines of $\mathbb{F}_2^3$, and obtain the group $\mathrm{PGL}(3,2)$, which is isomorphic to $\mathrm{GL}(3,2)$. Note that $\mathrm{PGL}(3,2)$ also permutes the hyperplanes of $\mathbb{F}_2^3$. Of course, an element of $\mathrm{PGL}(3,2)$ maps a line contained in a hyperplane to a line contained in a hyperplane, and so $\mathrm{PGL}(3,2)$ is contained in Aut(Hea), where Hea is the Heawood graph. Notice that $\mathrm{PGL}(3,2)$ is transitive on the lines of $\mathbb{F}_2^3$ and transitive on the hyperplanes of $\mathbb{F}_2^3$.

Now define $\tau : L \cup H \to L \cup H$ by $\tau\{\ell, h\} = \{h^\perp, \ell^\perp\}$. Note that τ is well-defined, as the subspace orthogonal to a line is a hyperplane, while the subspace orthogonal to a hyperplane is a line. Clearly $|\tau| = 2$ as $(s^\perp)^\perp = s$. In order to show that $\tau \in \mathrm{Aut(Hea)}$, let $\ell \in L$ and $h \in H$ such that $\ell \subset h$. Then every vector in $h^\perp$ is orthogonal to every vector in h, and as $\ell \subset h$, every vector in $h^\perp$ is orthogonal to every vector in ℓ. Thus $h^\perp \subset \ell^\perp$ and so if $\ell h \in E(\text{Hea})$, then $\tau(\ell h) \in E(\text{Hea})$. Thus $\tau \in \text{Aut}(H)$.

Lemma 2.4.1 *Let $g \in \text{GL}(n,q)$, and s a subspace of $\mathbb{F}_q^n$. Then $g(s^\perp)^\perp = (g^{-1})^T(s)$.*

PROOF. First recall that if $w, v \in \mathbb{F}_q^n$, then the dot product of w and v, $w \cdot v$, can also be written as $w^T v$, where for a matrix g, g^T denotes the transpose of g. Let $w_1, \ldots, w_r$ be a basis for $s^\perp$, so that $g(s^\perp)$ has basis $gw_1, \ldots, gw_r$. In order to show that $g(s^\perp)^\perp = (g^{-1})^T(s)$, it suffices to show that $(g^{-1})^T v$ is orthogonal to gw_i for any i and $v \in s$ as $\dim(s) + \dim(s^\perp) = n$. Then

$$(gw_i) \cdot (g^{-1})^T v = (gw_i)^T (g^{-1})^T v = w_i^T g^T (g^{-1})^T v = w_i^T v = 0.$$

□

Consider the canonical action of PGL(3,2) on $L \cup H$, so that $g \in \text{PGL}(3,2)$, then $g(\ell, h) = \{g(\ell), g(h)\}$. Now, let $g \in \text{PGL}(3,2)$, which will consider in the above action on $L \cup H$. Then

$$\begin{aligned} \tau^{-1} g \tau(\{\ell, h\}) &= \tau^{-1} g(\{h^\perp, \ell^\perp\}) \\ &= \tau^{-1}(\{g(h^\perp), g(\ell^\perp)\}) \\ &= \{g(\ell^\perp)^\perp, g(h^\perp)^\perp\} \\ &= \{(g^{-1})^T(\ell), (g^{-1})^T(h)\} \end{aligned}$$

Then $\tau^{-1} g \tau = g^{-1})^T$ so $\text{PGL}(3,2) \triangleleft \langle \text{PGL}(3,2), \tau \rangle$.

Now, $\langle \text{PGL}(3,2), \tau \rangle$ admits a complete block system $\mathscr{B}$ with 2 blocks of size 7. The subgroup of PGL(3,2) that stabilizes a line does not stabilize any hyperplane! So we have that PGL(3,2) acts inequivalently on the lines and hyperplanes. It can be shown using a theorem of Tutte that $\text{Aut}(\text{Hea}) = \langle \text{PGL}(3,2), \tau \rangle$.

2.5 The Embedding Theorem

Definition 2.5.1 Let Γ_1 and Γ_2 be digraphs. The **wreath product of** Γ_1 **and** Γ_2, denoted $\Gamma_1 \wr \Gamma_2$ is the digraph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and edge set

$$\{(u,v)(u,v') : u \in V(\Gamma_1) \text{ and } vv' \in E(\Gamma_2)\} \cup \{(u,v)(u',v') : uu' \in E(\Gamma_1) \text{ and } v, v' \in V(\Gamma_2)\}.$$

Intuitively, $\Gamma_1 \wr \Gamma_2$ is constructed as follows. First, we have $|V(\Gamma_1)|$ copies of the digraph Γ_2, with these $|V(\Gamma_1)|$ copies indexed by elements of $V(\Gamma_1)$. Next, between corresponding copies of Γ_2 we place every possible directed from one copy to another if in Γ_1 there is an edge between the indexing labels of the copies of Γ_2, and no edges otherwise.

To find the wreath product of any two graphs Γ_1 and Γ_2 (see Figure 2.3):

1. First corresponding to each vertex of Γ_1, put a copy of Γ_2.

2. If v_1 and v_2 are adjacent in Γ_1, put every edge between corresponding copies of Γ_2.

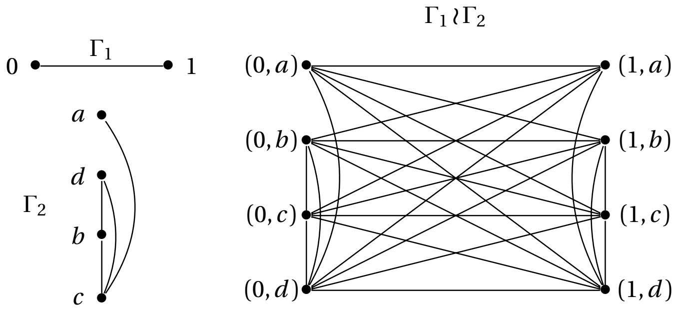

Figure 2.3: $\Gamma_1 \wr \Gamma_2$

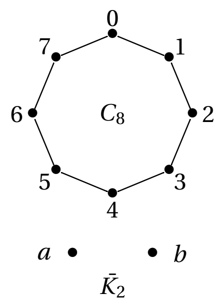

Figure 2.4: C_8

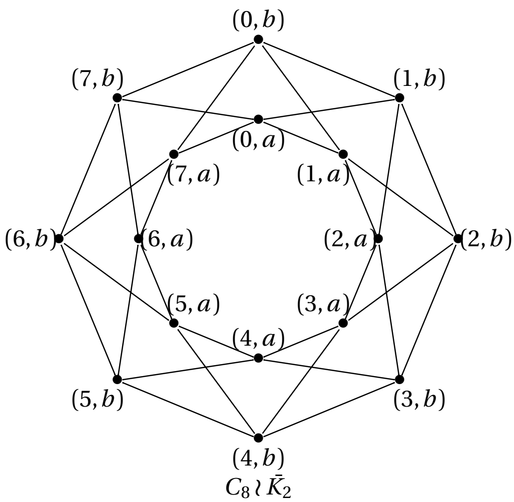

Figure 2.5: $C_8 \wr \bar{K}_2$

Let us consider the graph $C_8 \wr \bar{K}_2$ (see Figure 2.5).
In the previous graph, think of the sets $\{(i,j) : j \in \mathbb{Z}_2\}$ as blocks. Take any automor-

phism of C_8, and think of it as "permuting" the blocks. A block is mapped to a block by any automorphism of $\bar{K}_2$, and we can have different automorphisms of $\bar{K}_2$ for different blocks. This is the group $\mathrm{Aut}(C_8)\wr\mathrm{Aut}(\bar{K}_2)$.

Definition 2.5.2 Let G be a permutation group acting on X and H a permutation group acting on Y. Define the **wreath product of** G **and** H, denoted $G\wr H$, to be the set of all permutations of $X\times Y$ of the form $(x,y)\to(g(x),h_x(y))$.

Intuitively, the wreath product $G\wr H$ has elements of G permuting $|X|$ copies of Y, and as an element of G permutes these copies, the copies of Y are mapped to each via elements of H. Crucially, the elements of H chosen to map copies of Y mapped to each other are chosen independently.

Example 2.5.3 We show the group $\mathbb{Z}_p\wr\mathbb{Z}_p\le S_{p^2}$ has order p^{p+1} and consequently is a Sylow p-subgroup of S_{p^2}. As $\mathbb{Z}_p\wr\mathbb{Z}_p=\{(i,j)\mapsto(i+a,j+b_i):a,b_i\in\mathbb{Z}_p\}$, we see that $|\mathbb{Z}_p\wr\mathbb{Z}_p=p^{p+1}$ as there are p choices for a, as well as p choices for each of the p b_i. As the multiples of p dividing $p^2!$ are $p,2p,\ldots,(p-1)p,p^2$, we see that the largest power of p dividing $p^2!$ is p^{p+1}.

Definition 2.5.4 Let G be a transitive permutation group acting on X that admits a complete block system $\mathcal{B}$. Then the action of G on $\mathcal{B}$ induces a permutation group in $S_{\mathcal{B}}$, which we denote by $G/\mathcal{B}$. More specifically, if $g\in G$, then define $g/\mathcal{B}:\mathcal{B}\to\mathcal{B}$ by $g/\mathcal{B}(B)=B'$ if and only if $g(B)=B'$, and set $G/\mathcal{B}=\{g/\mathcal{B}:g\in G\}$.

Theorem 2.5.5 *Let G be a transitive permutation group acting on X that admits a complete block system $\mathcal{B}$. Then G is permutation isomorphic to a subgroup of*

$$(G/\mathcal{B})\wr(\mathrm{Stab}_G(\mathcal{B})|_{B_0}),$$

where $B_0\in\mathcal{B}$.

Proof. For each $B\in\mathcal{B}$, there exists h_B such that $h_B(B_0)=B$. Define $\lambda:X\to\mathcal{B}\times B_0$ by $\lambda(x)=(B,x_0)$, where $x\in B$ and $x_0=h_B^{-1}(x)$. Define $\phi:G\to(G/\mathcal{B})\wr(\mathrm{Stab}_G(\mathcal{B})|_{B_0})$ by $\phi(g)(B,x_0)=(g(B),h_{g(B)}^{-1}gh_B(x_0))$. We must show that λ is a bijection, ϕ is an injective homomorphism, and that $\lambda(g(x))=\phi(g)(\lambda(x))$ for all $x\in X$ and $g\in G$.

In order to show that λ is a bijection, it suffices to show that λ is one-to-one as by Theorem 2.2.2 it is certainly the case that $|X|=|\mathcal{B}\times B_0|$. Let $x,x'\in X$ and assume that $(B,x_0)=\lambda(x)=\lambda(x')$. Clearly then both x and x' are contained in B, and $x_0=h_B^{-1}(x)=h_B^{-1}(x')$. As h_B is a permutation, it follows that $x=x'$ and λ is one-to-one and so a bijection.

To show that ϕ is injective, suppose that $\phi(g)=\phi(g')$. Applying the definition of ϕ, we see that

$$(g(B),h_{g(B)}^{-1}gh_B(x_0))=(g'(B),h_{g'(B)}^{-1}g'h_B(x_0)),$$

for all $B\in\mathcal{B}$ and $x_0\in B_0$. It immediately follows that $g/\mathcal{B}=g'/\mathcal{B}$ and $h_{g(B)}^{-1}gh_B=h_{g'(B)}^{-1}g'h_B$. Using the fact that $g/\mathcal{B}=g'/\mathcal{B}$ and canceling, we see that $g=g'$ and ϕ is injective.

Let $g_1, g_2 \in G$. Then

$$\begin{aligned}
\phi(g_1)\phi(g_2)(B,x_0) &= \phi(g_1)(g_2(B), h_{g_2^{-1}(B)} g_2 h_B(x_0)) \\
&= (g_1 g_2(B), h^{-1}_{g_1(g_2(B))} g_1 h_{g_2(B)}(h^{-1}_{g_2(B)} g_2 h_B(x_0)) \\
&= (g_1 g_2(B), h^{-1}_{g_1 g_2(B)} g_1 g_2 h_B(x_0)) \\
&= \phi(g_1 g_2)(B, x_0),
\end{aligned}$$

and so ϕ is a homomorphism.

Finally, observe that $\phi(g)(\lambda(x)) = \phi(g)(B,x_0) = (g(B), h^{-1}_{g(B)} g h_B(x_0))$ while

$$\lambda(g(x)) = (g(B), h^{-1}_{g(B)} g(x)) = (g(B), h^{-1}_{g(B)} g h_B(x_0)),$$

and so $\lambda(g(x)) = \phi(g)(\lambda(x))$ for all $x \in X$ and $g \in G$. □

The following immediate corollary is often useful.

Corollary 2.5.6 *Let G be a transitive permutation group that admits a complete block system $\mathcal{B}$ consisting of m blocks of size k. Then G is permutation isomorphic to a subgroup of $S_m \wr S_k$.*

One must be slightly careful with this labeling, as it is not always the most natural labeling. For example, let q and p be prime with $q|(p-1)$ and $\alpha \in \mathbb{Z}_p^*$ of order q. Define $\rho, \tau : \mathbb{Z}_q \times \mathbb{Z}_p \mapsto \mathbb{Z}_q \times \mathbb{Z}_p$ by $\tau(i,j) = (i+1, \alpha j)$ and $\rho(i,j) = (i, j+1)$. Then $\langle \rho, \tau \rangle$ is isomorphic to the nonabelian group of order qp. The labeling that one would get for this group by applying the Embedding Theorem is $\langle \rho', \tau' \rangle$, where $\rho'(i,j) = (i, j+\alpha^i)$, $\tau'(i,j) = (i+1, j)$.

Exercise 2.5.7 *Draw the graph $K_4 \wr \bar{K}_3$.*

Exercise 2.5.8 *Show that $G \wr H$ has order $|G| \cdot (|H|)^{|X|}$, where G acts on X.*

Exercise 2.5.9 *Show that a Sylow p-subgroup of S_{p^k}, $k \geq 1$ is $\mathbb{Z}_p \wr \mathbb{Z}_p \wr \cdots \wr \mathbb{Z}_p$, where the wreath product is taken k times.*

Exercise 2.5.10 *Verify that the graph wreath product is associative.*

Exercise 2.5.11 *Verify that the permutation group wreath product is associative.*

Exercise 2.5.12 *Show that* $\mathrm{Aut}(\bar{\Gamma}_1 \wr \bar{\Gamma}_2) = \mathrm{Aut}(\overline{\Gamma_1 \wr \Gamma_2})$.

Exercise 2.5.13 *For vertex-transitive graphs* Γ_1 *and* Γ_2, *show that* $\mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2) \leq \mathrm{Aut}(\Gamma)$.

2.6 A Graph Theoretic Tool

Let G be a transitive group that admits a normal complete block system $\mathcal{B}$ consisting of m blocks of prime size p. Then $\mathrm{fix}_G(\mathcal{B})|_B$ is a transitive group of prime degree p, and so contains a p-cycle. Define a relation $\equiv$ on $\mathcal{B}$ by $B \equiv B'$ if and only if whenever $\gamma \in \mathrm{fix}_G(\mathcal{B})$ then $\gamma|_B$ is a p-cycle if and only if $\gamma|_{B'}$ is also a p-cycle (here $\gamma|_B$ is the induced permutation of g on B). It is straightforward to verify that $\equiv$ is an equivalence relation (Exercise 2.6.2). Let C be an equivalence class of $\equiv$ and $E_C = \cup_{B\in C} B$ (remember that the equivalence classes of $\equiv$ consist of *blocks* of $\mathcal{B}$), and $\mathcal{E} = \{E_C : C \text{ is an equivalence class of } C\}$.

Lemma 2.6.1 *Let* Γ *be a digraph with* $G \le \mathrm{Aut}(\Gamma)$ *admit a normal complete block system* $\mathcal{B}$ *consisting of* m *blocks of prime size* p*. Let* $\equiv$ *and* $\mathcal{E}$ *be defined as in the preceding paragraph. Then* $\mathcal{E}$ *is a complete block system of* G *and for every* $g \in \mathrm{fix}_G(\mathcal{B})$*,* $g|_E \in \mathrm{Aut}(\Gamma)$ *for every* $E \in \mathcal{E}$*. Here* $g|_E(x) = g(x)$ *if* $x \in E$ *while* $g(x) = x$ *if* $x \notin E$*.*

Proof. We will first show that $\mathcal{E}/\mathcal{B}$ is a complete block system of $G/\mathcal{B}$ by showing that $\equiv$ is a $G/\mathcal{B}$-congruence and applying Theorem 2.2.9. This will then imply that $\mathcal{E}$ is a complete block system of G. We thus need to show that if $B \equiv B'$ and $g \in G$, then $g(B) \equiv g(B')$ for every $g \in G$. Suppose that $g(B) \not\equiv g(B')$. Then there exists $\gamma \in \mathrm{fix}_G(\mathcal{B})$ such that $\gamma|_{g(B)}|_B$ is a p-cycle but $\gamma|_{g(B')}$ is not a p-cycle. Let $b \in B$. Then $g^{-1}\gamma g(b) = g^{-1}\gamma(g(b))$ and so $g^{-1}\gamma g|_B$ is a p-cycle, while a similar argument shows that $g^{-1}\gamma g|_B$ is not. We conclude that if $B \equiv B'$ then $g(B) \equiv g(B')$ and so $\mathcal{E}$ is indeed a complete block system of G.

Now suppose that $B \not\equiv B'$. We will first show that in Γ, either every vertex of B is our or in adjacent to every vertex of B', or there is no edge between any vertex of B and any vertex of B'. So, suppose that there is an edge from say B to B'. As $B \not\equiv B'$, there is $\gamma \in \mathrm{fix}_G(\mathcal{B})$ such that $\gamma|_B$ is a p-cycle while $\gamma|_{B'}$ is not a p-cycle. Raising γ to the power $|\gamma|_{B'}|$ which is relatively prime to γ, we may assume without loss of generality that $\gamma|_{B'} = 1$. Let the directed edge $\overrightarrow{b_0b'} \in E(\Gamma)$, where $b_0 \in B$ and $b' \in B'$. As $\gamma|_B$ is a p-cycle, we may write $\gamma|_B = (b_0\ b_1\ \dots\ b_{p-1})$ (i.e. we are writing $\gamma|_B$ as a p-cycle starting at b_0). Applying γ to the edge $\overrightarrow{b_0b'}$, we obtain the edge $\overrightarrow{b_1b'}$, and applying γ to the edge $\overrightarrow{b_0b'}$ r times, we obtain the edge $\overrightarrow{b_rb'}$. We conclude that $\overrightarrow{bb'} \in E(\Gamma)$ for every $b \in B$. Now, there exists $\delta \in \mathrm{fix}_G(\mathcal{B})$ such that $\delta|_{B'}$ is a p-cycle. Applying δ to each of the edges $\overrightarrow{bb'}$ $p-1$ times (similar to above), we have that the edges $\overrightarrow{bb'} \in E(\Gamma)$ for every $b \in B$ and $b' \in \mathcal{B}$. Similar arguments will show that if $\overrightarrow{b'b} \in E(\Gamma)$ for some $b' \in B'$ and $b \in B$, then $\overrightarrow{b'b} \in E(\Gamma)$ for every $b' \in B'$ and $b \in B$, as well as if $bb' \in E(\Gamma)$ for some $b, b' \in E(\Gamma)$, then $bb' \in E(\Gamma)$ for all $b \in B$ and $b' \in B'$.

Now, let $\gamma \in \mathrm{fix}_G(\mathcal{B})$, and consider the map $\gamma|_E$, $E \in \mathcal{E}$. If $e = \overrightarrow{xy} \in E(\Gamma)$ and both $x, y \in E$, then surely $\gamma|_E(e) = \gamma(e) \in E(\Gamma)$. Similarly, if both $x, y \notin E$, then $\gamma|_E(e) = e \in E(\Gamma)$. If $x \in E$ but $y \notin E(\Gamma)$, then let $B_x, B_y \in \mathcal{B}$ such that $x \in B_x$ and $y \in B_y$. Then $\overrightarrow{x'y'} \in E(\Gamma)$ for every $x' \in B_x$, $y' \in B_y$ by arguments above. Also, $\gamma(x) = x' \in B_x$, and so $\gamma|_E(e) = \overrightarrow{x'y} \in E(\Gamma)$. An analogous argument will show that $\gamma|_E(e) \in E(\Gamma)$ if $x \notin E$ but $y \in E$. As in every case, $\gamma|_E \in E(\Gamma)$, we have that $\gamma|_E \in \mathrm{Aut}(\Gamma)$ establishing the result. $\square$

The above result also holds in the more general situation that $\mathrm{fix}_G(\mathcal{B})$ acts primitively on $B \in \mathcal{B}$.

Exercise 2.6.2 *Write a careful proof that* $\equiv$ *is an equivalence relation.*

2.7 Basic Definitions Concerning Graphs

Definition 2.7.1 Let G be a group and $S \subset G$. Define a **Cayley digraph of** G, denoted $\mathrm{Cay}(G,S)$ to be the graph with $V(\mathrm{Cay}(G,S)) = G$ and $E(\mathrm{Cay}(G,S)) = \{(g,gs) : g \in G, s \in S\}$. We call S the **connection set of** $\mathrm{Cay}(G,S)$.

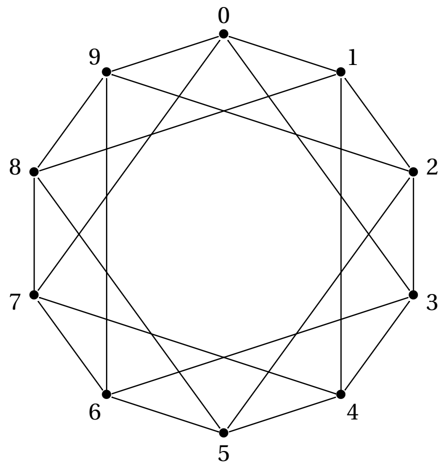

Figure 2.6: The Cayley graph $\mathrm{Cay}(\mathbb{Z}_{10}, \{1,3,7,9\})$.

If we additionally insist that $S = S^{-1} = \{s^{-1} : s \in S\}$ (or if the group is abelian and the operation is addition, that $S = -S$), then there will be no directed edges in $\mathrm{Cay}(G,S)$, and we obtain a **Cayley graph**. This follows as if $(g,gs) \in E(\mathrm{Cay}(G,S))$ and $s^{-1} \in S$, then $(gs, gs(s^{-1})) = (gs,s) \in E(\mathrm{Cay}(G,S))$. In many situations, whether or not a Cayley digraph has loops doesn't have any effect. In these cases the default is usually to exclude loops by also insisting that $1_G \notin S$ (or $0 \notin S$ if G is abelian and the operation is addition).

Perhaps the most common Cayley digraphs that one encounters are Cayley digraphs of the cyclic groups $\mathbb{Z}_n$ of order n, as in Figure 2.6. A Cayley (di)graph of $\mathbb{Z}_n$ is called a **circulant** (di)graph of order n.

Definition 2.7.2 For a group G, the **left regular representation** , denoted G_L , is the subgroup of S_G given by the left translations of G. More specifically, $G_L = \{x \to gx : g \in G\}$. We denote the map $x \to gx$ by g_L. It is straightforward to verify that G_L is a group and that $G_L \cong G$.

Let $x, y \in G$, and $g = yx^{-1}$. Then $g_L(x) = yx^{-1}x = y$ so that G_L is transitive on G.

Lemma 2.7.3 *For every* $S \subseteq G$, $G_L \le \mathrm{Aut}(\mathrm{Cay}(G,S))$.

PROOF. Let $e = (g, gs) \in E(\mathrm{Cay}(G,S))$, where $g \in G$ and $s \in S$. Let $h \in G$. We must show that $h_L(e) \in E(\mathrm{Cay}(G,S))$, or that $h_L(e) = (g', g's')$ for some $g' \in G$ and $s' \in S$. Setting $g' = hg$ and $s' = s$, we have that

$$h_L(e) = h_L(g, gs) = (hg, h(gs)) = (hg, (hg)s) = (g', g's').$$

□

In general, for an abelian group G, the group G_L will consist of "translations by g" that map $x \to x+g$. That is, $G_L = \{x \to x+g : g \in G\}$. More specifically, for a cyclic group $\mathbb{Z}_n$, we have that $\mathbb{Z}_n$ is generated by the map $x \to x+1$ (or course instead on 1, one could put any generator of $\mathbb{Z}_n$).

The following important result of G. Sabidussi [**?**] characterizes Cayley graphs.

Theorem 2.7.4 *A graph Γ is isomorphic to a Cayley graph of a group G if and only if* $\mathrm{Aut}(\Gamma)$ *contains a regular subgroup isomorphic to G.*

PROOF. If $\Gamma \cong \mathrm{Cay}(G,S)$ with $\phi : \Gamma \to \mathrm{Cay}(G,S)$ an isomorphism, then by Lemma 2.7.3, $\mathrm{Aut}(\mathrm{Cay}(G,s))$ contains the regular subgroup $G_L \cong G$, namely $\phi^{-1}G_L\phi$ (see Exercise 2.7.6). Conversely, suppose that $\mathrm{Aut}(\Gamma)$ contains a regular subgroup $H \cong G$, with $\omega : H \to G$ an isomorphism. Fix $v \in V(\Gamma)$. As H is regular, for each $u \in V(\Gamma)$, there exists a unique $h_u \in H$ such that $h_u(v) = u$. Define $\phi : V(\Gamma) \to G$ by $\phi(u) = \omega(h_u)$. Note that as each h_u is unique, ϕ is well-defined and is also a bijection as ω is a bijection. Let $U = \{u \in V(\Gamma) : (v,u) \in E(\Gamma)\}$. We claim that $\phi(\Gamma) = \mathrm{Cay}(G, \phi(U))$.

As $\phi(V(\Gamma)) = G$, $V(\phi(\Gamma)) = G$. Let $e \in E(\phi(\Gamma))$. We must show that $e = (g, gs)$ for some $g \in G$ and $s \in \phi(U)$. As $e \in E(\phi(\Gamma))$, $\phi^{-1}(e) = (u_1, u_2) \in E(\Gamma)$ by Exercise 2.7.6. Let $w \in V(\Gamma)$ such that $h_{u_1}(w) = u_2$. Then $h_{u_1}^{-1}(u_1, u_2) = (v, w)$ so that $w = h_w(v) \in U$, and $h_{u_2} = h_{u_1}h_w$ as $h_{u_1}h_w(v) = h_{u_1}(w) = u_2$. Thus

$$(u_1, u_2) = (h_{u_1}(v), h_{u_1}(w)) = (h_{u_1}(v), h_{u_1}h_w(v)) = (h_{u_1}(v), h_{u_2}(v)).$$

Set $g = \omega(h_{u_1})$ and $s = \omega(h_w)$. Then

$$\phi(u_1, u_2) = (\omega(h_{u_1}), \omega(h_{u_2})) = (\omega(h_{u_1}), \omega(h_{u_1}h_w)) = (\omega(h_{u_1}), \omega(h_{u_1})\omega(h_w)) = (g, gs)$$

as required. □

We now prove a well-known result first proven by Turner [**?**].

Theorem 2.7.5 *Every transitive group of prime degree p contains a cyclic regular subgroup. Consequently, every vertex-transitive digraph is isomorphic to a circulant digraph of order p.*

PROOF. Let G be a transitive group of prime degree p. As G is transitive, it has one orbit of size p, and so p divides $|G|$. Hence G has an element of order p, which is necessarily a p-cycle permuting all of the points. So G contains a regular cyclic subgroup, and the result follows by Theorem 2.7.4. □

Exercise 2.7.6 *Show that if $\phi : \Gamma \to \Gamma'$ is a graph isomorphism, then $\phi^{-1} : \Gamma' \to \Gamma$ is also a graph isomorphism. Then show that if $H \le \mathrm{Aut}(\Gamma')$, then $\phi^{-1} H \phi \le \mathrm{Aut}(\Gamma)$.*

Exercise 2.7.7 *Show that $(\mathbb{Z}_m)_L \wr (\mathbb{Z}_n)_L$ contains a regular subgroup isomorphic to $\mathbb{Z}_{mn}$. Consequently, the wreath product of two circulant digraphs is a circulant digraph.*

Exercise 2.7.8 *Show that for any two groups G and H, $G_L \wr H_L$ contains a regular subgroup isomorphic to $G \times H$. Deduce that the wreath product of two Cayley digraphs is a Cayley digraph.*

2.8 An Application to Graphs

Definition 2.8.1 Let m and n be positive integers, and $\alpha \in \mathbb{Z}_n^*$. Define $\rho, \tau : \mathbb{Z}_m \times \mathbb{Z}_n \mapsto \mathbb{Z}_m \times \mathbb{Z}_n$ by $\rho(i,j) = (i, j+1)$ and $\tau(i,j) = (i+1, \alpha j)$. A vertex-transitive Γ digraph with vertex set $\mathbb{Z}_m \times \mathbb{Z}_n$ is an (m,n)-**metacirculant digraph** if and only if $\langle \rho, \tau \rangle \le \mathrm{Aut}(\Gamma)$.

The Petersen graph is a $(2,5)$-metacirculant graph with $\alpha = 2$, while the Heawood graph is a $(2,7)$-metacirculant graph with $\alpha = 6$.

Lemma 2.8.2 *Let $\rho : \mathbb{Z}_m \times \mathbb{Z}_n \mapsto \mathbb{Z}_m \times \mathbb{Z}_n$ by $\rho(i,j) = (i, j+1)$. Then $Z_{S_{mn}}(\langle \rho \rangle) = \{(i,j) \mapsto (\sigma(i), j + b_i) : \sigma \in S_n, b_i \in \mathbb{Z}_n\} = S_m \wr (\mathbb{Z}_n)_L$.*

Proof. Straightforward computations will show that every element of $\{(i,j) \mapsto (\sigma(i), j + b_i)$ does indeed centralize $\langle \rho \rangle$. Then $Z_{S_{mn}}(\langle \rho \rangle)$ is transitive as $\rho \in Z_{S_{mn}}(\langle \rho \rangle)$. Additionally, $\langle \rho \rangle \triangleleft Z_{S_{mn}}(\langle \rho \rangle)$, and so the orbits $\mathcal{B}$ of $\langle \rho \rangle$ form a complete block system of $Z_{S_{mn}}(\langle \rho \rangle)$. Let $B \in \mathcal{B}$, and $g \in \mathrm{Stab}_{Z_{S_{mn}}(\langle \rho \rangle)}(B)$. Then $g|_B$ commutes with $\langle \rho \rangle|_B$, and as $\langle \rho \rangle|_B$ is a regular cyclic group, it is self-centralizing (we have already seen that a transitive abelian group is regular in the proof of Theorem 2.2.5. The subgroup generated by any element that centralizes a regular abelian group and the regular abelian group is a transitive abelian group, and so regular.) Then $\mathrm{Stab}_{Z_{S_{mn}}(\langle \rho \rangle)}(B)|_B \le \langle \rho \rangle|_B$, and so by the Embedding Theorem 2.5.5, $Z_{S_{mn}}(\langle \rho \rangle) \le S_m \wr (\mathbb{Z}_n)_L$. As $S_m \wr (\mathbb{Z}_n)_L \le Z_{S_{mn}}(\langle \rho \rangle)$, the result follows. $\square$

Theorem 2.8.3 *A vertex-transitive digraph Γ of order qp, q and p distinct primes, is isomorphic to a (q,p)-metacirculant digraph if and only if $\mathrm{Aut}(\Gamma)$ has a transitive subgroup G that contains a normal complete block system $\mathcal{B}$ with q blocks of size p.*

Proof. If Γ is isomorphic to a (q,p)-metacirculant, then after an appropriate relabeling, $\langle \rho, \tau \rangle \le \mathrm{Aut}(\Gamma)$. Then $\langle \rho \rangle \triangleleft \langle \rho, \tau \rangle = G$ has orbits of length p.

Conversely, suppose that there exists $N \triangleleft G \le \mathrm{Aut}(\Gamma)$ and N has orbits of length p. Let $\mathcal{B}$ be the complete block system formed by the orbits of N, and assume that G is the largest subgroup of $\mathrm{Aut}(\Gamma)$ that admits $\mathcal{B}$. Then $G/\mathcal{B}$ is transitive, and so G contains an element τ such that $\langle \tau \rangle / \mathcal{B}$ is cyclic of order q (and so regular), and τ has order a power of q. By Lemma 2.6.1 there exists $\rho \in G$ such that $\langle \rho \rangle$ is semiregular of order p, and a Sylow p-subgroup P of $\mathrm{fix}_G(\mathcal{B})$ has order p or p^q. If $|P| = p^q$, then if there is a directed edge in Γ from some vertex of B to some vertex of B', $B, B' \in \mathcal{B}$, then there is a directed edge from every vertex of B to every vertex of B'. We conclude that Γ is isomorphic to a wreath product of vertex-transitive digraphs of order q and p, respectively, and so by Theorem 2.7.5, Γ is isomorphic to the wreath product of a circulant digraph of order q and a circulant digraph of order p. By Exercise 2.8.4, Γ is isomorphic to a Cayley digraph of $\mathbb{Z}_q \times \mathbb{Z}_p$, and every such digraph is isomorphic to a (q,p)-metacirculant digraph by Exercise 2.8.5. We henceforth assume that $|P| = p$.

Now, $\langle\rho\rangle$ and $\tau^{-1}\langle\rho\rangle\tau$ are contained in Sylow p-subgroups P_1 and P_2 of $\mathrm{fix}_G(\mathcal{B})$, respectively, and so there exists $\delta \in G$ such that $\delta^{-1}P_2\delta = P_1$. Replacing τ with $\tau\delta$, if necessary, we assume without loss of generality that $\tau^{-1}\langle\rho\rangle\tau \le P_1$. As $|P_1| = |P_2| = p$, we see that $\tau^{-1}\langle\rho\rangle\tau = \langle\rho\rangle$. We now label the vertex set of Γ with elements of $\mathbb{Z}_q \times \mathbb{Z}_p$ in such a way that $\rho(i,j) = (i, j+1)$, and $\tau(i,j) = (i+1, \omega_i(j))$, where $\omega_i \in S_p$, $i \in \mathbb{Z}_q$. Set $\tau^{-1}\rho\tau = \rho^a$, where $a \in \mathbb{Z}_p^*$. Define $\bar{a} : \mathbb{Z}_q \times \mathbb{Z}_p \to \mathbb{Z}_q \times \mathbb{Z}_p$ by $\bar{a}(i,j) = (i, aj)$. Then $\bar{a}^{-1}\rho^a\bar{a} = \rho$. Then $\tau\bar{a}$ centralizes $\langle\rho\rangle$, and so by Lemma 2.8.2, we see that $\tau\bar{a} \in \{(i,j) \mapsto (\sigma(i), j+b_i) : \sigma \in S_q, b_i \in \mathbb{Z}_p\}$. We conclude that $\tau\bar{a}(i,j) = (i+1, j+b_i)$, $b_i \in \mathbb{Z}_p$, and so $\tau(i,j) = (i+1, a^{-1}j + c_i)$, $c_i \in \mathbb{Z}_p$.

Let $H = \langle \tau, z_k : k \in \mathbb{Z}_q\rangle$, where $z_k(i,j) = (i, j+\delta_{ik})$, where δ_{ik} is Kronecker's delta function. That is $\delta_{ik} = 1$ if $i = k$ and 0 otherwise. Note that $\langle z_k : k \in \mathbb{Z}_q\rangle \triangleleft H$ and $H/\langle z_k : k \in \mathbb{Z}_q\rangle \cong \langle\tau\rangle$. We conclude that $\langle\tau\rangle$ is a Sylow q-subgroup of H. Now let, $\tau' : \mathbb{Z}_q \times \mathbb{Z}_p \mapsto \mathbb{Z}_q \times \mathbb{Z}_p$ by $\tau'(i,j) = (i+1, a^{-1}j)$. Then $\tau' \in H$ and also has order $|\tau|$, and so $\langle\tau'\rangle$ is a Sylow q-subgroup of H as well. Thus there exists $\gamma \in H$ such that $\gamma^{-1}\langle\tau\rangle\gamma = \langle\tau'\rangle$. Also, $\langle\rho\rangle \triangleleft H$, and so $\gamma^{-1}\langle\rho\rangle\gamma = \langle\rho\rangle$, and so $\gamma^{-1}\langle\rho,\tau\rangle\gamma = \langle\rho,\tau'\rangle$. Then Γ is isomorphic to a (q,p)-metacirculant digraph. □

Exercise 2.8.4 *Show that for any two groups G and H, $G_L \wr H_L$ contains a regular subgroup isomorphic to $G \times H$. Deduce that the wreath product of two Cayley digraphs is a Cayley digraph.*

Exercise 2.8.5 *Let n be a positive integer and $n = mk$, where* $\gcd(m,k) = 1$. *Show that any circulant digraph of order n is isomorphic to an (m,k)-metacirculant digraph.*

2.9 A General Strategy for Analyzing Imprimitive Permutation Groups with Blocks of Prime Size - Especially Automorphism Groups of Vertex-transitive Digraphs

Let G be a transitive group that admits a complete block system $\mathcal{B}$ with blocks of prime size p. If $\mathcal{B}$ is not a normal complete block system, then $G/\mathcal{B}$ is a transitive faithful representation of G, so hopefully one can use induction... Otherwise, $\mathcal{B}$ is normal. If $\mathrm{fix}_G(\mathcal{B})$ is not faithful on $B \in \mathcal{B}$, then in the general case, one cannot say much about $\mathrm{fix}_G(\mathcal{B})$ other than the normalizer of a Sylow p-subgroup of $\mathrm{fix}_G(\mathcal{B})$ is a vector space invariant under its normalizer, which is transitive. Tools from linear algebra may be employed - not promising but not hopeless either. In the case of the automorphism group of a vertex-transitive graph, one may employ Lemma 2.6.1 in which case the Sylow p-subgroup of $\mathrm{fix}_G(\mathcal{B})$ has a very restrictive structure as we have seen. If $\mathrm{fix}_G(\mathcal{B})$ is faithful on $B \in \mathcal{B}$ there are three cases to consider. The first is when $\mathrm{fix}_G(\mathcal{B}) \cong \mathbb{Z}_p$. This is the most difficult case to deal with, and nothing more will be said of this case now. If $\mathrm{fix}_G(\mathcal{B}) \neq \mathbb{Z}_p$, then there are two subcases, depending on whether or not the action of $\mathrm{fix}_G(\mathcal{B})$ on $B, B' \in \mathcal{B}$ are always equivalent or if the are inequivalent. We now investigate this...

Lemma 2.9.1 *Let $G \le S_n$ be transitive on V and admit a normal complete block system $\mathcal{B}$ with blocks of size p. Suppose that $\mathrm{fix}_G(\mathcal{B}) \neq \mathbb{Z}_p$ is faithful on $B \in \mathcal{B}$. Define an equivalence relation $\equiv$ on V by $v_1 \equiv v_2$ if and only if $\mathrm{Stab}_{\mathrm{fix}_G(\mathcal{B})}(v_1) = \mathrm{Stab}_{\mathrm{fix}_G(\mathcal{B})}(v_2)$. Then the equivalence classes of $\equiv$ are blocks of G, and each equivalence class of $\equiv$ contains at most one point of $B \in \mathcal{B}$.*

Proof. As conjugation by an element of G maps the stabilizer of a point in $\mathrm{fix}_G(\mathcal{B})$ to the stabilizer of a point in $\mathrm{fix}_G(\mathcal{B})$, $\equiv$ is a G-congruence, and so by Theorem 2.2.9 the equivalence classes of $\equiv$ are blocks of G. If a block contains two points from the same equivalence class, then by first part of this lemma applied to $\mathrm{fix}_G(\mathcal{B})|_B$, we see that $\mathrm{fix}_G(\mathcal{B})|_B$ is imprimitive. But a transitive group of prime degree is primitive, a contradiction. □

Let $\mathcal{E}$ be the complete block system consisting of the equivalence classes of $\equiv$ in the previous lemma. Suppose that each equivalence class of $\equiv$ contains exactly one element of each $B \in \mathcal{B}$. This means that $|B \cap E| = 1$ for every $B \in \mathcal{B}$ and $E \in \mathcal{E}$. Two complete $\mathcal{B}$ and $\mathcal{C}$ of G such that $|B \cap C| = 1$ for every $B \in \mathcal{B}$ and $C \in \mathcal{C}$ are called **orthogonal complete block systems**. Observe that if $\mathcal{B}$ and $\mathcal{C}$ are orthogonal and $\mathcal{B}$ consists of m blocks of size k, then $\mathcal{C}$ consists of k blocks of size m.

Lemma 2.9.2 *Let $n = mk$ and $G \le S_n$ such that G is transitive and admits orthogonal complete block systems $\mathcal{B}$ and $\mathcal{C}$ of m blocks of size k and k blocks of size m, respectively. Then G is permutation equivalent to a subgroup of $S_k \times S_m$ in its natural action on $\mathbb{Z}_k \times \mathbb{Z}_m$.*

Proof. Note that G has a natural action on $\mathcal{B} \times \mathcal{C}$ given by $g(B,C) = (g(B), g(C))$, and that in this action each $g \in G$ induces a permutation contained in $S_{\mathcal{B}} \times S_{\mathcal{C}}$, namely, $(g/\mathcal{B}, g/\mathcal{C})$. Any element of G in the kernel of this representation of G must fix every block of $\mathcal{B}$ and every block of $\mathcal{C}$. As $|B \cap C| = 1$ for every $B \in \mathcal{B}$ and $C \in \mathcal{C}$, and there are exactly $mk = n$ such intersections, the kernel of this representation is the identity and the representation is faithful. Let $B \in \mathcal{B}$ and $C \in \mathcal{C}$. If $g \in G$ stabilizes the point (B,C) in this representation, then $g(B) = B$ and $g(C) = C$. Let $B \cap C = \{x\}$. Then $g(x) = x$. Conversely, if $g(x) = x$, then there exists $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $x \in B$ and $x \in \mathcal{C}$. Then $g(B,C) = (B,C)$ so $\mathrm{Stab}_G(x) = \mathrm{Stab}_G((B,C))$. It then follows by Theorem 2.3.2 that these two actions of G are equivalent. □

Combining the two previous lemmas we have:

Lemma 2.9.3 *Let $G \le S_n$ be transitive on V and admit a normal complete block system $\mathcal{B}$ with blocks of size p. Suppose that $\mathrm{fix}_G(\mathcal{B}) \neq \mathbb{Z}_p$ is faithful on $B \in \mathcal{B}$. If the action of $\mathrm{fix}_G(\mathcal{B})$ on B and B' are always equivalent, then G is permutation isomorphic to a subgroup of $S_{n/p} \times S_p$.*

We now illustrate these techniques by calculating the full automorphism group of circulant digraphs of order p^2, where p is prime.

Theorem 2.9.4 *Let Γ be a circulant digraph of order p^2, p an odd prime. Then one of the following is true:*

- $(\mathbb{Z}_{p^2})_L \triangleleft \mathrm{Aut}(\Gamma)$, *or*
- $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2)$ *where, Γ_1 and Γ_2 are circulant digraphs of prime order, or*
- $\Gamma = K_{p^2}$ *or its complement and* $\mathrm{Aut}(\Gamma) = S_{p^2}$.

Note: The result is simpler if $p = 2$, and as $|S_4| = 24$, everything can be easily determined by hand.

PROOF. A **Burnside group** is a group G with the property that whenever $H \le S_n$ contains G as a regular subgroup, then either H is doubly-transitive or H is imprimitive. Here, H is doubly-transitive if whenever we have two order pairs (x_1, y_1) and (x_2, y_2) with $x_1 \ne y_1$ and $x_2 \ne y_2$, then there exists $h \in H$ such that $h(x_1, y_1) = (x_2, y_2)$. Schur showed that a cyclic group of composite order is a Burnside group [**?**, Theorem 3.5A]. So $\mathrm{Aut}(\Gamma)$ is either imprimitive or doubly-transitive. If $\mathrm{Aut}(\Gamma)$ is doubly-transitive, then Γ is either K_{p^2} or its complement and the result follows. Otherwise, $\mathrm{Aut}(\Gamma)$ admits a complete block system $\mathcal{B}$ consisting of p blocks of size p. (In the case $\mathbb{Z}_q \times \mathbb{Z}_p$, we still have a Burnside group, while for $\mathbb{Z}_p \times \mathbb{Z}_p$, the possibilities for a simply primitive group are given explicitly by the O'Nan-Scott Theorem.)

Let $\rho : \mathbb{Z}_{p^2} \mapsto \mathbb{Z}_{p^2}$ by $\rho(i) = i+1 \pmod{p^2}$, so that $\langle\rho\rangle$ is a regular subgroup of $\mathrm{Aut}(\Gamma)$ of order p^2. $\mathcal{B}$ is then necessarily normal, and formed by the orbits of a normal subgroup of $\langle\rho\rangle$ of order p. There is a unique such subgroup, namely $\langle\rho^p\rangle$. Consider the equivalence relation $\equiv$ on $\mathcal{B}$ by $B \equiv B'$ if and only if whenever $\gamma \in \mathrm{fix}_G(\mathcal{B})$ then $\gamma|_B$ is a p-cycle if and only if $\gamma|_{B'}$ is also a p-cycle. By Lemma 2.6.1, the union of the equivalence classes of $\equiv$ form a complete block system $\mathcal{E}$, and $\rho^p|_E \in \mathrm{Aut}(\Gamma)$ for every $E \in \mathcal{E}$. If $\mathcal{E}$ has blocks of size p, then $\mathcal{E} = \mathcal{B}$, and Γ is isomorphic to the wreath product of two circulant digraphs of prime order. A result of Sabidussi [**?**] then gives (2).

If $\mathcal{E}$ consists of one block of size p^2, then $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ acts faithfully on $B \in \mathcal{B}$ as otherwise, as a normal subgroup of a primitive group is necessarily transitive, the kernel K of the action of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ on $B \in \mathcal{B}$ is transitive on some $B' \in \mathcal{B}$, and so K has order divisible by p. Then K contains an element which is a p-cycle on B' and the identity on B, and so $B \not\equiv B'$. We first consider when $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B}) \not\cong \mathbb{Z}_p$.

We now wish to apply a famous result of Burnside which states that a transitive group of prime degree is either permutation isomorphic to a subgroup of $\mathrm{AGL}(1,p) = \{x \mapsto ax + b : a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$, or is a doubly-transitive group with nonabelian socle. A consequence of the Classification of the Finite Simple groups is that all doubly-transitive groups are known [**?**, Table], and then one can show (by examining each possible case), that a doubly-transitive group either has 1 or 2 inequivalent representations. If $H \le \mathrm{AGL}(1,p)$ is transitive and not isomorphic to $\mathbb{Z}_p$ (note that p will divide $|H|$), then $|H| = ap$, $a > 1$. Then $\mathrm{Stab}_H(x)$ has order a, and as $\mathrm{AGL}(1,p)$ is solvable of order $p(p-1)$, H is solvable and $\gcd(a,p) = 1$. By Hall's Theorem, any two subgroups of order a are conjugate in H. We conclude by Theorem 2.3.2 that H has a unique representation of degree p.

Now define an equivalence relation $\equiv'$ on $\mathbb{Z}_{p^2}$ by $i \equiv j$ if and only if $\mathrm{Stab}_{\mathrm{fix}_{\mathrm{Aut}(\Gamma)}}(i) = \mathrm{Stab}_{\mathrm{fix}_{\mathrm{Aut}(\Gamma)}}(j)$. Note that as $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ is primitive on $B \in \mathcal{B}$, no equivalence class of $\equiv'$ can contain more than one element of $B \in \mathcal{B}$. If there is a unique representation of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ as a transitive group of degree p, then the equivalence classes of $\equiv'$ form an orthogonal complete block system of $\mathrm{Aut}(\Gamma)$. However, as $\mathbb{Z}_{p^2}$ contains a unique subgroup of order p and every complete block system is normal, there is no such orthogonal complete block system, a contradiction! (Note that if $\mathbb{Z}_{p^2}$ is replaced with $\mathbb{Z}_p \times \mathbb{Z}_p$ of $\mathbb{Z}_q \times \mathbb{Z}_p$, there is no contradiction, but we still are done as then $\mathrm{Aut}(\Gamma)$ is contained in a direct product and it is easy to figure out what happens). We thus assume that $\mathrm{fix}_G(\mathcal{B}) \cong \mathbb{Z}_p$.

Of course $\mathrm{Aut}(\Gamma)/\mathcal{B}$ is a transitive group of prime degree, so by Burnside's Theorem it is either contained in $\mathrm{AGL}(1,p)$ or is a doubly-transitive group with nonabelian socle. If $\mathrm{Aut}(\Gamma)/\mathcal{B} \leq \mathrm{AGL}(1,p)$, then as $\mathrm{AGL}(1,p)$ contains a normal Sylow p-subgroup which is necessarily $\langle\rho\rangle/\mathcal{B}$, we see that $\langle\rho\rangle \triangleleft \mathrm{Aut}(\Gamma)$ and the result follows. I will not really talk about the other case - it doesn't really have much to do with imprimitive groups, and is also the hardest case. I will say that in the case under consideration, this cannot occur, while if $q \neq p$ it not only can occur, but in fact has two different outcomes. For $\mathbb{Z}_p \times \mathbb{Z}_p$ it also can occur but only has the obvious outcome of being something like $H \times \mathbb{Z}_p$, where $H \leq S_p$. □

2.10 Further Reading

Extensions of Burnside's Theorem to transitive groups of degree p^2 as well as the full automorphism groups of all vertex-transitive digraphs of order p^2 can be found in [**?**]. An extension of Burnside's Theorem for transitive groups that contain a regular cyclic group of prime-power order can be found in [**?**]. An extension of Burnside's Theorem for groups that contain a regular abelian Hall π-subgroup is in [**?**]. Some information about transitive groups of degree qp can be found in [**?**], together with the full automorphism groups of all vertex-transitive graphs of order qp. An extension of Burnside's Theorem for a regular semidirect product of two cyclic groups of prime-power degree can be found in [**?**] (we remark that while the results in this paper are stated only for graphs, the graph structure is not used much - so the result is not explicitly stated, but can be extracted).

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Chapter 3

Leonard pairs and the q-Racah polynomials

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SUMMARY

A dual system of orthogonal polynomials arises from the Bose-Mesner algebra of an association scheme that is metric and cometric (or P- and Q-polynomial in Delsarte-Bannai's term). D. Leonard classified such orthogonal polynomials and they turned out to be in one to one correspondence with the q-Racah polynomials or certain limits of these polynomials.

P. Terwilliger interpreted the above dual system of orthogonal polynomials as two linear transformations each acting in an irreducible tridiagonal fashion on an eigenbasis of the other one. He called such linear transformations a Leonard pair. He classified them and established a representation theory for them. This immediately leads to a classification of dual systems of orthogonal polynomials as well as a characterization of the q-Racah polynomials including their limiting cases.

The theme of my lectures will be Terwilliger's theory of Leonard pairs. After a brief introduction to the relation between a Leonard pair and a dual system of orthogonal polynomials, I will introduce a raising map R and a lowering map L via the split decomposition (weight-space decomposition) attached to a Leonard pair. The eigenvalues of the Leonard pair will be written explicitly by the Askey-Wilson parameters. The key here is the tridiagonal relations (TD-relations). It turns out that the TD-relations nearly characterize Leonard pairs.

I will then introduce pre-Leonard pairs, relaxing the conditions for Leonard pairs, and ask when a pre-Leonard pair is a Leonard pair. The keys here are Terwilliger's lemma and the Askey-Wilson relations. We show that a pre-Leonard pair is a Leonard pair if and only if the data of the pre-Leonard pair, i.e., the eigenvalues together with the local traces of LR, allow Askey-Wilson parametrizations. This estasblishes a bijection between the set of data and the isomorphism classes of Leonard pairs. It also gives a way to construct a Leonard pair for each admissible data.

Finally I will explicitly write down dual systems of orthogonal polynomials as q-Racah polynomials and explain the limiting cases.

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Leonard pairs

and

the q-Racah polynomials

— Terwilliger's theory

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Leonard pairs and the q-Racah polynomials

— Terwilliger's theory

1. Background in algebraic combinatorics
2. L-pairs and dual system of orthogonal polynomials
3. TD-pairs : weight space decomposition
4. TD-pairs : TD-relations and Terwilliger's lemma
5. Classification of L-pairs
6. Classification of dual systems of orth. polynomials

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1st lecture Background in algebraic combinatorics

$\Gamma = (X, R)$ simple graph

X : point set

R : edge set

$R \subset X \times X$ no multiple edges

$R = {}^tR = \{(x,y) \mid (y,x) \in R\}$

undirected graph

$R \cap \Delta = \phi$

no loops

$\Delta = \{(x,x) \mid x \in X\}$

$|X| < \infty$ finite graph

$\partial(x,y)$ = the length of shortest paths joining x and y

Assume Γ is connected.

$d = \mathrm{Max}\{\partial(x,y) \mid x,y \in X\} < \infty$

the diameter

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§1.1

Examples of 'nice' graphs

(1) Johnson graph $J(v,k)$, $v \geq 2k$

V : finite set, $|V| = v$

$X = \binom{V}{k}$: the set of k-subsets of V

$R \ni (x,y) \iff |x \cap y| = k-1$

$d = k$ (diameter)

(2) q-Johnson graph $J_q(v,k)$, $v \geq 2k$

V : v-dim vector space $/\mathbb{F}_q$

$X = \binom{V}{k}$: the set of k-dim subspaces of V

$R \ni (x,y) \iff \dim x \cap y = k-1$

$d = k$ (diameter)

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(3) Hamming graph $H(n, q)$

F : finite set, $|F| = q$

$X = F^n = F \times F \times \cdots \times F \ni x = (x_1, x_2, \cdots, x_n)$
$y = (y_1, y_2, \cdots, y_n)$

$R \ni (x, y) \iff \# \{ i \mid x_i \neq y_i \} = 1$

$d = n$ (diameter)

(4) Bilinear forms graph $Bil_{m \times n}(q)$

X : the set of $m \times n$ matrices $/ \mathbb{F}_q$

$R \ni (x, y) \iff \mathrm{rank}(x - y) = 1$

$d = \mathrm{Min}\{m, n\}$ (diameter)

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§1.2

How nice ?

$\Gamma = (X, R)$: finite, connected simple graph
diameter d

A : the adjacency matrix of Γ

$$A(x, y) = \begin{cases} 1 & \text{if } (x, y) \in R \\ 0 & \text{otherwise} \end{cases}$$

A_i : ith distance matrix

$$A_i(x, y) = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{otherwise} \end{cases}$$

In particular,

$A = A_1$

1st nice property

$\exists\, v_i(x) \in \mathbb{R}[x]$ polynomial of degree i

s.t. $A_i = v_i(A)$

distance-regular, P-polynomial

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$V = \mathbb{R}^X = \{ f : X \longrightarrow \mathbb{R} \} \simeq \mathbb{R}^n, \quad |X| = n$

$M_X(\mathbb{R}) = \{ B : X \times X \longrightarrow \mathbb{R} \} \simeq M_n(\mathbb{R})$

$M_X(\mathbb{R})$ acts on $V = \mathbb{R}^X$

by $\displaystyle (Bf)(x) = \sum_{y \in X} B(x,y) f(y)$

$$M_X(\mathbb{R}) \supset \mathfrak{A} = \langle \underset{\substack{\| \\ I}}{A_0}, \underset{\substack{\| \\ A}}{A_1}, A_2, \cdots, A_d \rangle$$

<u>Bose - Mesner algebra</u>

$\dim \mathfrak{A} = d+1$

$V = V_0 + V_1 + \cdots + V_d$ eigenspace decomp. of A

θ_i : eigenvalue of A on V_i

$E_i : V \longrightarrow V_i$ projection

$A = \theta_0 E_0 + \theta_1 E_1 + \cdots + \theta_d E_d$

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$M_X(\mathbb{R}) \supset \mathfrak{A} = \langle A_0, A_1, \cdots, A_d \rangle$
$= \langle E_0, E_1, \cdots, E_d \rangle$

$$\begin{cases} I = E_0 + E_1 + \cdots + E_d & \text{primitive idempotents} \\ E_i E_j = \delta_{ij} E_i \end{cases}$$

$M_X(\mathbb{R})^\circ$: the algebra $M_X(\mathbb{R})$ w.r.t. the entry-wise product $\circ$
(<u>Hadamard product</u>, Schur - product)

$(B_1 \circ B_2)(x,y) = B_1(x,y) B_2(x,y)$

$M_X(\mathbb{R})^\circ \supset \mathfrak{A}^\circ = \langle E_0, E_1, \cdots, E_d \rangle$ <u>dual BM-alg.</u>
$= \langle A_0, A_1, \cdots, A_d \rangle$

$$\begin{cases} J = A_0 + A_1 + \cdots + A_d \\ \text{all one matrix (the identity of } M_X(\mathbb{R})^\circ) \\ A_i \circ A_j = \delta_{ij} A_i \end{cases}$$

<u>2nd nice property</u>

$\exists\, v_i^*(x) \in \mathbb{R}[x]$ polynomial of degree i

s.t. $n E_i = v_i^*(n E_1)$, $n = |X|$

<u>Q- polynomial</u> in $\mathfrak{A}^\circ$.

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§1.3

P- and Q- polynomial scheme

$\Gamma(X, R)$: finite, connected simple graph
diameter d, $|X| = n$

$$M_X(\mathbb{R}) \supset \mathfrak{A} = \langle \underset{\substack{\| \\ I}}{A_0}, \underset{\substack{\| \\ A}}{A_1}, \cdots, A_d \rangle \quad \text{BM-alg.}$$

$$= \langle E_0, E_1, \cdots, E_d \rangle \quad \text{primitive idempotents}$$

$$A = A_1 = \theta_0 E_0 + \theta_1 E_1 + \cdots + \theta_d E_d$$

$$M_X(\mathbb{R})^\circ \supset \mathfrak{A}^\circ = \langle \underset{\substack{\| \\ J}}{nE_0}, \underset{\substack{\| \\ nE}}{nE_1}, \cdots, nE_d \rangle \quad \text{dual BM-alg}$$

$$= \langle A_0, A_1, \cdots, A_d \rangle$$

$$nE = nE_1 = \theta_0^* A_0 + \theta_1^* A_1 + \cdots + \theta_d^* A_d$$

$\exists$ $v_i(x)$, $v_i^*(x) \in \mathbb{R}[x]$ polynomials of degree i $(0 \le i \le d)$

s.t.

$$A_i = v_i(A)$$

$$nE_i = v_i^*(nE)$$

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Fact

$\Gamma(X, R)$: P- and Q-polynomial

$\Longrightarrow$

$\{v_i(x)\}_{i=0}^d$: orthogonal polynomial with support $\{\theta_0, \theta_1, \cdots, \theta_d\}$

$\{v_i^*(x)\}_{i=0}^d$: orthogonal polynomial with support $\{\theta_0^*, \theta_1^*, \cdots, \theta_d^*\}$

and

$$\frac{v_i(\theta_j)}{v_i(\theta_0)} = \frac{v_j^*(\theta_i^*)}{v_j^*(\theta_0^*)} \quad \text{all } i, j$$

dual system of orthogonal polynomials

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Leonard Theorem

$\{v_i\}_{i=0}^{d}$, $\{v_i^*\}_{i=0}^{d}$: dual system of orth. polys.

$\Longrightarrow$

$$\left\{\frac{1}{v_i(\theta_0)} v_i(x)\right\}_{i=0}^{d}, \quad \left\{\frac{1}{v_i^*(\theta_0^*)} v_i^*(x)\right\}_{i=0}^{d}$$

the q-Racah polynomials
(Askey - Wilson polynomials)

or
their limits

Leonard 1982

Bannai - Ito 1984

Terwilliger 2001
in terms of Leonard pairs

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§1.4

Terwilliger algebra

$\Gamma = (X, R)$ finite connected simple graph
P- and Q-polynomial

$X \ni x_0$ base point fixed

$X_i = X_i(x_0) = \{x \in X \mid \partial(x_0 x) = i\}$

$V_i^* = V_i^*(x_0) = \{f : X \longrightarrow \mathbb{C} \mid f(x) = 0 \text{ if } x \notin X_i\}$

Then

$V = \mathbb{C}^X = V_0^* \oplus V_1^* \oplus \cdots \oplus V_d^*$

$E_i^* = E_i^*(x_0) : V \longrightarrow V_i^*$ projection

$\mathcal{T}^* = \mathcal{T}^*(x_0) = \langle E_0^*, E_1^*, \cdots, E_d^* \rangle \subset M_X(\mathbb{C}) \simeq End(V)$

$$\begin{cases} I = E_0^* + E_1^* + \cdots + E_d^* \\ E_i^* E_j^* = \delta_{ij} E_i^* \end{cases}$$ primitive idempotents

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$$\mathfrak{A} = \langle A_0, A_1, \cdots, A_d \rangle \subset M_X(\mathbb{C}), \qquad \text{BM-alg}$$
$$= \langle E_0, E_1, \cdots, E_d \rangle$$

$$\mathfrak{A}^\circ = \langle nE_0, nE_1, \cdots, nE_d \rangle \subset M_X(\mathbb{C})^\circ, \qquad \text{dual BM-alg}$$
$$= \langle A_0, A_1, \cdots, A_d \rangle \qquad n = |X|$$

$$\begin{cases} J = A_0 + A_1 + \cdots + A_d \\ A_i \circ A_j = \delta_{ij} A_i \end{cases}$$

$$nE = nE_1 = \theta_0^* A_0 + \theta_1^* A_1 + \cdots + \theta_d^* A_d$$

$$\begin{array}{ccccc} M_X(\mathbb{C})^\circ & & M_X(\mathbb{C}) & & \\ \cup & & \cup & & \\ \mathfrak{A}^\circ & \xrightarrow{\sim} & \mathfrak{A}^* & & \text{dual BM-alg} \\ A_i & \longmapsto & E_i^* & & \\ nE & \longmapsto & A^* & = & \theta_0^* E_0^* + \theta_1^* E_1^* + \cdots + \theta_d^* E_d^* \\ nE_i = v_i^*(nE) & \longmapsto & A_i^* & = & v_i^*(A^*) \end{array}$$

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$$T = T(x_0) = \langle \mathfrak{A}, \mathfrak{A}^* \rangle \subset M_X(\mathbb{C})$$

Terwilliger algebra

$$T = \langle A, A^* \rangle$$

$$A_i = v_i(A) \qquad (A = A_1)$$
$$A_i^* = v_i^*(A^*) \qquad (A^* = A_1^*)$$

$$E_i : \quad V = \bigoplus_{j=0}^{d} V_j \longrightarrow V_i \qquad \text{projection}$$
ith eigenspace of A

$$E_i^* : \quad V = \bigoplus_{j=0}^{d} V_j^* \longrightarrow V_i^* \qquad \text{projection}$$
ith eigenspace of A^*

Proposition

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \qquad 0 \le i \le d \quad (V_{-1}^* = V_{d+1}^* = 0)$$

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, \qquad 0 \le i \le d \quad (V_{-1} = V_{d+1} = 0)$$

More precisely

$$E_j^* A E_i^* \begin{cases} = 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases}$$

$$E_j A^* E_i \begin{cases} = 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases}$$

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<u>Proposition</u>

$T = T(x_0)$ is a semi-simple algebra.

In particular

$V = \mathbb{C}^X$ is a direct sum of irreducible T-submodules

<u>Corollary</u>

$V = \mathbb{C}^X \supset W$: irreducible T-submodule

$\Longrightarrow$

$A|_W,\ A^*|_W \in End(W)$

are a <u>TD-pair</u>.

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<u>TD-pair</u> (tridiagonal pair)

W : finite dim. vector space $/\mathbb{C}$

$A, A^* \in End(W)$: diagonalizable

$W = \bigoplus_{i=0}^{d} W_i$ eigenspace decomp. of A

$W = \bigoplus_{i=0}^{d^*} W_i^*$ eigenspace decomp. of A^*

The pair A, A^* is called a TD-pair if

(i) $A W_i^* \subseteq W_{i-1}^* + W_i^* + W_{i+1}^*$, $0 \le i \le d^*$ $(W_{-1}^* = W_{d^*+1}^* = 0)$

$A^* W_i \subseteq W_{i-1} + W_i + W_{i+1}$, $0 \le i \le d$ $(W_{-1} = W_{d+1} = 0)$

and

(ii) W is irreducible as an $\langle A, A^* \rangle$-module.

<u>Remark</u> $d = d^*$

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Def

$A, A^* \in \mathrm{End}(W)$: L-pair (Leonard pair)

$\Longleftrightarrow$

TD-pair

and

$\dim V_i = \dim V_i^* = 1, \quad 0 \le i \le d.$

Example

$T = T(x_0) = \langle A, A^* \rangle \subset \mathrm{End}(V), \quad V = \mathbb{C}^X$

Terwilliger alg. of a P- and Q-poly. scheme

Set

$W = \{ f \in V \mid f \text{ is constant on each } X_i \},$

where

$X_i = \{ x \in X \mid \partial(x_0, x) = i \}.$

Then W is an irreducible T-module

and $\dim W \cap V_i = 1, \quad 0 \le i \le d$

$\dim W \cap V_i^* = 1, \quad 0 \le i \le d.$

W is called the principal T-module

(~~primary T-module~~)

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2nd lecture   L-pairs and dual system of orth. polys.

§2.1 orthogonal polynomials $/\mathbb{C}$ with finite support

$u_0(x) = 1, \; u_1(x), \; \cdots\cdots, \; u_d(x) \in \mathbb{C}[x]$

$\deg u_i(x) = i, \quad 0 \le i \le d$

orthogonal polynomials with support $\theta_0, \theta_1, \cdots, \theta_d \in \mathbb{C}$

$(\theta_i \ne \theta_j)$

if $\displaystyle\sum_{\nu=0}^{d} u_i(\theta_\nu) u_j(\theta_\nu) \mu_\nu = \frac{1}{k_i} \delta_{ij}, \quad$ all i, j

for some $\mu_0, \mu_1, \cdots, \mu_d \in \mathbb{C}$

$\mu_\nu \ne 0, \; 0 \le \nu \le d, \quad \displaystyle\sum_{\nu=0}^{d} \mu_\nu = 1$

and $k_0, k_1, \cdots, k_d \in \mathbb{C}$

$k_i \ne 0, \; 0 \le i \le d, \quad k_0 = 1.$

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$\{u_i(x)\}_{i=0}^{d}$ orth. polys. with support $\{\theta_\nu\}_{\nu=0}^{d}$

$\Longrightarrow$

(*) $x u_i(x) = c_i u_{i-1}(x) + a_i u_i(x) + b_i u_{i+1}(x), \quad 0 \le i \le d$

where $u_{-1}(x) = 0, \quad b_d = 1$

$$u_{d+1}(x) = \frac{1}{b_0 b_1 \cdots b_{d-1}} (x-\theta_0)(x-\theta_1)\cdots(x-\theta_d)$$

$$B = \begin{pmatrix} a_0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{d-1} & a_{d-1} & b_{d-1} \\ 0 & & & c_d & a_d \end{pmatrix}$$

<u>tridiagonal</u> matrix

<u>irreducible</u>

$c_i \neq 0, \ b_i \neq 0$ all i

<u>diagonalizable</u>

with eigenvalues $\{\theta_\nu\}_{\nu=0}^{d}$

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Conversely

B : tridiagonal matrix
irreducible
diagonalizable with eigenvalues $\{\theta_\nu\}_{\nu=0}^{d}$

$\Longrightarrow$

$\{u_i(x)\}_{i=0}^{d}$ defined by (*)

orth. polys. with support $\{\theta_\nu\}_{\nu=0}^{d}$

$$k_0 = 1, \quad k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}, \quad 0 \le i \le d$$

$$\mu_\nu = \frac{1}{k_d \, u'_{d+1}(\theta_\nu)\, u_d(\theta_\nu)}, \quad 0 \le \nu \le d$$

Christoffel number

<u>Remark</u>

B : tridiagonal, irreducible, diagonalizable

$\Longrightarrow$

Each eigenvalue of B has multiplicity 1.

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$\mathcal{P} = \{ \text{orth. polys. with finite support} \}$

$\mathcal{M} = \left\{ \begin{array}{l} \text{tridiagonal matrices that are irreducible} \\ \text{and diagonalizable} \end{array} \right\}$

$$\mathcal{P} \xleftrightarrow{1:1} \mathcal{M}$$

$\mathcal{P} \supset \mathcal{P}(\theta_0) = \{ \{u_i(x)\} \in \mathcal{P} \mid u_i(\theta_0) = 1 \quad \text{all } i \}$

$\mathcal{M} \supset \mathcal{M}(\theta_0) = \{ B \in \mathcal{M} \mid \text{row sum of } B = \theta_0 \}$

$$\mathcal{P}(\theta_0) \xleftrightarrow{1:1} \mathcal{M}(\theta_0)$$

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§2.2 dual systems of orth. polys.
and L-pairs

$\{u_i(x)\}_{i=0}^{d}$ orth. polys. with support $\{\theta_\nu\}_{\nu=0}^{d}$

$\{u_i^*(x)\}_{i=0}^{d}$ orth. polys. with support $\{\theta_\nu^*\}_{\nu=0}^{d}$

<u>dual</u>

iff $\exists$ ordering $\theta_0, \theta_1, \cdots, \theta_d$
$\exists$ ordering $\theta_0^*, \theta_1^*, \cdots, \theta_d^*$

$$u_j(\theta_i) = u_i^*(\theta_j^*) \qquad \text{all } i, j$$

<u>Remark</u>

$u_j(\theta_0) = u_0^*(\theta_j^*) = 1 \qquad \text{all } j$

$u_i^*(\theta_0^*) = u_0(\theta_i) = 1 \qquad \text{all } i$

$\{u_i(x)\}_{i=0}^{d} \in \mathcal{P}(\theta_0), \qquad \{u_i^*(x)\}_{i=0}^{d} \in \mathcal{P}(\theta_0^*)$

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$\mathcal{P}(\theta_0) \ni \{u_i(x)\}_{i=0}^{d} \longleftrightarrow B \in \mathcal{M}(\theta_0)$

$\mathcal{P}(\theta_0^*) \ni \{u_i^*(x)\}_{i=0}^{d} \longleftrightarrow B^* \in \mathcal{M}(\theta_0^*)$

<u>Proposition</u>

$\{u_i(x)\}_{i=0}^{d}$, $\{u_i^*(x)\}_{i=0}^{d}$ are dual

$\Longleftrightarrow$

$\exists\, S$ non singular matrix

$$S^{-1} B S = \begin{pmatrix} \theta_0 & & 0 \\ & \ddots & \\ 0 & & \theta_d \end{pmatrix} = D \quad \text{diagonal matrix}$$

$$S B^* S^{-1} = \begin{pmatrix} \theta_0^* & & 0 \\ & \ddots & \\ 0 & & \theta_d^* \end{pmatrix} = D^* \quad \text{diagonal matrix}$$

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$B,\ D^* \in M_{d+1}(\mathbb{C}) \simeq End(V)$, $\quad V = \mathbb{C}^{d+1}$

eigenspace decomposition of B

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_d, \qquad \dim V_i = 1, \quad 0 \le i \le d$$

$$V_i = \langle \mathbf{s}_i \rangle, \qquad B\mathbf{s}_i = \theta_i \mathbf{s}_i$$

$$S = (\mathbf{s}_0, \mathbf{s}_1, \cdots, \mathbf{s}_d)$$

eigenspace decomposition of D^*

$$V = V_0^* \oplus V_1^* \oplus \cdots \oplus V_d^*, \qquad \dim V_i^* = 1, \quad 0 \le i \le d$$

$$V_i^* = \langle \mathbf{e}_i \rangle, \qquad D^* \mathbf{e}_i = \theta_i^* \mathbf{e}_i$$

$$\mathbf{e}_i = \begin{array}{cc} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} & \begin{matrix} \\ \\ (i \\ \\ \\ \end{matrix} \end{array}$$

(i) $B V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$, $\quad 0 \le i \le d$, $\quad V_{-1}^* = V_{d+1}^* = 0$

B : tridiagonal

$D^* V_i \subseteq V_{i-1} + V_i + V_{i+1}$, $\quad 0 \le i \le d$, $\quad V_{-1} = V_{d+1} = 0$

$B^* = S^{-1} D^* S$: tridiagonal

(ii) V : irreducible as a $\langle B, D^* \rangle$-module

B : irreducible tridiagonal

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$A,\ A^* \in End(V)$ diagonalizable, $V \simeq \mathbb{C}^{d+1}$

$V = \bigoplus_{i=0}^{d} V_i$, $\dim V_i = 1$, $0 \le i \le d$

$A|_{V_i} = \theta_i \in \mathbb{C}$, $\theta_i \ne \theta_j$

eigenspace decomposition of A

$V = \bigoplus_{i=0}^{d} V_i^*$, $\dim V_i^* = 1$, $0 \le i \le d$

$A^*|_{V_i^*} = \theta_i^* \in \mathbb{C}$, $\theta_i^* \ne \theta_j^*$

eigenspace decomposition of A^*

<u>L-pair</u> (Leonard pair)

if $\exists$ ordering $\theta_0, \theta_1, \cdots, \theta_d$

$\exists$ ordering $\theta_0^*, \theta_1^*, \cdots, \theta_d^*$

$$\begin{cases} A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, & 0 \le i \le d, \quad V_{-1}^* = V_{d+1}^* = 0 \\ A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, & 0 \le i \le d, \quad V_{-1} = V_{d+1} = 0 \end{cases}$$

and

V is irreducible as an $\langle A, A^* \rangle$-module.

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$A_1,\ A_1^* \in End(V)$ L-pair

$A_2,\ A_2^* \in End(V)$ L-pair

$(A_1, A_1^*) \simeq (A_2, A_2^*)$ isomorphic

if $\exists\ \phi : V \longrightarrow V$ isomorphism as vector spaces

$$\begin{tikzcd} V \arrow[r, "{A_1, A_1^*}"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, "\phi"] \\ V \arrow[r, "{A_2, A_2^*}"'] & V \end{tikzcd}$$

$A_2 = \phi A_1 \phi^{-1}$

$A_2^* = \phi A_2 \phi^{-1}$

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$\mathcal{P}(\theta_0) \ni \{u_i(x)\}_{i=0}^{d} \longleftrightarrow B \in \mathcal{M}(\theta_0)$

$\mathcal{P}(\theta_0^*) \ni \{u_i^*(x)\}_{i=0}^{d} \longleftrightarrow B^* \in \mathcal{M}(\theta_0^*)$

$\{u_i(x)\}_{i=0}^{d}, \quad \{u_i^*(x)\}_{i=0}^{d} \quad : \quad$ dual

$\Longleftrightarrow$

$B, \ D^* \quad : \quad$ L-pair

In this case,

$(B, D^*) \simeq (D, B^*)$ isomorphic as L-pairs

Remark

$A, A^* \in End(V)$ L-pair

Then

$\exists$ a basis of V, the matrix B of A $\in \mathcal{M}(\theta_0)$ the matrix D^* of A^* is diagonal

and

$\exists$ a basis of V, the matrix D of A is diagonal the matrix B^* of A^* $\in \mathcal{M}(\theta_0^*)$

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§2.3 Leonard's theorem — classification of dual systems of orth. polys.

$(a)_i = a(a+1)\cdots(a+i-1), \quad i = 1, 2, \cdots$

$(a)_0 = 1$ <u>shifted factorial</u>

$${}_{r+1}F_r\left(\begin{matrix} a_1, \cdots, a_{r+1} \\ b_1, \cdots, b_r \end{matrix}; x\right) = \sum_{i=0}^{\infty} \frac{(a_1)_i \cdots\cdots (a_{r+1})_i}{(b_1)_i \cdots\cdots (b_r)_i} \frac{x^i}{i!}$$

<u>hypergeometric series</u>

$(a;q)_i = (1-a)(1-aq)\cdots(1-aq^{i-1}), \quad i = 1, 2, \cdots$

$(a;q)_0 = 1$ <u>q-shifted factorial</u>

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, \cdots, a_{r+1} \\ b_1, \cdots, b_r \end{matrix}; q, x\right) = \sum_{i=0}^{\infty} \frac{(a_1;q)_i \cdots\cdots (a_{r+1};q)_i}{(b_1;q)_i \cdots\cdots (b_r;q)_i} \frac{x^i}{(q;q)_i}$$

<u>basic hypergeometric series</u>

(q-hypergeometric series)

$q \longrightarrow 1, \quad {}_{r+1}\phi_r \longrightarrow {}_{r+1}F_r$

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<u>Theorem</u> (Leonard 1982, Bannai - Ito 1984)

$\{u_i(x)\}_{i=0}^d$, $\{u_i^*(x)\}_{i=0}^d$: a dual system of orthogonal polynomials with support $\{\theta_\nu\}_{\nu=0}^d$, $\{\theta_\nu^*\}_{\nu=0}^d$, respectively

$\Longleftrightarrow$

$\{u_i(x)\}_{i=0}^d$, $\{u_i^*(x)\}_{i=0}^d$: a dual system of q-Racah polynomials or its limiting case

$$u_i(x) = {}_4\phi_3\left(\begin{matrix} q^{-i}, & s^*q^{i+1}, & q^{-\nu}, & sq^{\nu+1} \\ q^{-d}, & r_1q, & r_2q & \end{matrix} ; q, q\right)$$

$$x = \xi(\nu) = \theta_0 + h\frac{1}{q^\nu}(1-q^\nu)(1-sq^{\nu+1})$$

$$\theta_i = \xi(i), \quad 0 \le i \le d$$

$$u_i^*(x) = {}_4\phi_3\left(\begin{matrix} q^{-i}, & sq^{i+1}, & q^{-\nu}, & s^*q^{\nu+1} \\ q^{-d}, & r_1q, & r_2q & \end{matrix} ; q, q\right)$$

$$x = \xi^*(\nu) = \theta_0^* + h^*\frac{1}{q^\nu}(1-q^\nu)(1-s^*q^{\nu+1})$$

$$\theta_i^* = \xi^*(\nu), \quad 0 \le i \le d$$

(details in 6th lecture)

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§2.4 Terwilliger's theorem — Classification of L-pairs

start with

$A, A^* \in End(V)$, $V \simeq \mathbb{C}^{d+1}$

L - pair

$$V = \bigoplus_{i=0}^d V_i \qquad \text{e. s. decomp. of } A, \quad \dim V_i = 1$$

$$= \bigoplus_{i=0}^d V_i^* \qquad \text{e. s. decomp. of } A^*, \quad \dim V_i^* = 1$$

$$\begin{cases} AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, & V_{-1}^* = V_{d+1}^* = 0 \\ A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}, & V_{-1} = V_{d+1} = 0 \end{cases}$$

Set

$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d), \quad 0 \le i \le d$$

<u>weight space</u>

Then

$$V = \bigoplus_{i=0}^d U_i \qquad \dim U_i = 1, \quad 0 \le i \le d$$

<u>weight space decomposition</u>

(split decomposition)

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$F_i : V \longrightarrow U_i$ projection

$R = A - \sum_{i=0}^{d} \theta_i F_i$, $\quad \theta_i$: eigenvalue of A on V_i

$L = A^* - \sum_{i=0}^{d} \theta_i^* F_i$, $\quad \theta_i^*$: eigenvalue of A^* on V_i^*

Then

$$\begin{cases} R\, U_i = U_{i+1}, & 0 \le i \le d, \quad U_{d+1} = 0 \\ L\, U_i = U_{i-1}, & 0 \le i \le d. \quad U_{-1} = 0 \end{cases}$$

R <u>raises</u> $\{U_i\}_{i=0}^{d}$

L <u>lowers</u> $\{U_i\}_{i=0}^{d}$

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Conversely, start with

$V = \bigoplus_{i=0}^{d} U_i$, $\quad \dim U_i = 1$

$F_i : V \longrightarrow U_i$ projection.

Given

$$\begin{cases} \theta_0, \theta_1, \cdots, \theta_d \in \mathbb{C}, & \theta_i \ne \theta_j \\ \theta_0^*, \theta_1^*, \cdots, \theta_d^* \in \mathbb{C}, & \theta_i^* \ne \theta_j^* \end{cases}$$

and

$R, L \in \mathrm{End}(V)$ raising, lowering maps

$$\begin{cases} R\, U_i = U_{i+1}, & 0 \le i \le d, \quad U_{d+1} = 0 \\ L\, U_i = U_{i-1}, & 0 \le i \le d, \quad U_{-1} = 0 \end{cases}$$

Set

$$\begin{cases} A = R + \sum_{i=0}^{d} \theta_i F_i \\ A^* = L + \sum_{i=0}^{d} \theta_i^* F_i . \end{cases}$$

Such $A, A^* \in \mathrm{End}(V)$ are called a <u>pre L-pair</u>.

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Choose $u_0 \in U_0$, $u_0 \neq 0$

Set $u_i = R^i u_0$.

Then $U_i = \mathbb{C} u_i$, $0 \le i \le d$

and $L u_{i+1} = \lambda_i u_i$, $0 \le i \le d-1$

for some $\lambda_i \in \mathbb{C}^\times = \mathbb{C} - \{0\}$.

The <u>data</u> $\left(\{\theta_i\}_{i=0}^{d}, \{\theta_i^*\}_{i=0}^{d}, \{\lambda_i\}_{i=0}^{d-1} \right)$

determines the isomorphism class of a pre L-pair $A, A^* \in \mathrm{End}(V)$.

If $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$, $i \neq j \in \{0, 1, \cdots, d\}$

and $\lambda_i \neq 0$, $0 \le i \le d-1$,

then $\left(\{\theta_i\}_{i=0}^{d}, \{\theta_i^*\}_{i=0}^{d}, \{\lambda_i\}_{i=0}^{d-1} \right)$

is the data of some pre L-pair A, A^*.

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<u>Theorem</u> (Terwilliger 2001)

$A, A^* \in \mathrm{End}(V)$ pre L-pair

with data $\left(\{\theta_i\}_{i=0}^{d}, \{\theta_i^*\}_{i=0}^{d}, \{\lambda_i\}_{i=0}^{d-1} \right)$.

Then

A, A^* are an L-pair

$\Longleftrightarrow$

$$\theta_i = \theta_0 + h \frac{1}{q^i} (1-q^i)(1-sq^{i+1}), \quad 0 \le i \le d$$

$$\theta_i^* = \theta_0^* + h^* \frac{1}{q^i} (1-q^i)(1-s^*q^{i+1}), \quad 0 \le i \le d$$

$$\lambda_i = hh^* q^{-2i-1} (1-q^{i+1})(1-q^{i-d})(1-r_1 q^{i+1})(1-r_2 q^{i+1}), \quad 0 \le i \le d-1$$

for some $r_1, r_2, s, s^*, h, h^*, \theta_0, \theta_0^*$ and $q \in \mathbb{C}$

$r_1 r_2 = s s^* q^{d+1}$

$h h^* \neq 0$, $q^i \neq 1$, $1 \le i \le d$

$s, s^* \notin \{q^{-2}, q^{-3}, \cdots, q^{-2d}\}$

$r_1, r_2 \notin \{q^{-1}, q^{-2}, \cdots, q^{-d}\} \cup \{s^*q, s^*q^2, \cdots, s^*q^d\}$ if $s^* \neq 0$

$r_2 = 0$, $r_1 \notin \{q^{-1}, q^{-2}, \cdots, q^{-d}\} \cup \{sq, sq^2, \cdots, sq^d\}$ if $s^* = 0$

or the limiting case

(details in 5th lecture)

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3rd lecture

§3 TD-pairs : weight space decomposition

V: finite dim vector space $/\mathbb{C}$

$A, A^* \in End(V)$ diagonalizable

$V = \bigoplus_{i=0}^{d} V_i$ eigenspace decomp. of A

$V = \bigoplus_{i=0}^{d^*} V_i^*$ eigenspace decomp. of A^*

A, A^* : TD-pair (tridiagonal pair)

if

(i) $\exists$ standard ordering $V_0, V_1, \cdots, V_d$

$\exists$ standard ordering $V_0^*, V_1^*, \cdots, V_{d^*}^*$

$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \le i \le d^*, \quad V_{-1}^* = V_{d^*+1}^* = 0$

$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, \quad 0 \le i \le d, \quad V_{-1} = V_{d+1} = 0$

and

(ii) V is irreducible as an $\langle A, A^* \rangle$- module

$V \supseteq W$ A-inv., A^*-inv. subspace

$\Longrightarrow W = V$ or 0.

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Remark

(1) $d = d^*$ holds. diameter

trivial TD-pair if $d = 0$.

We assume $d \ge 1$ unless otherwise stated.

(2) $A, A^* \in End(V)$: L-pair (Leonard pair)

if $\dim V_i = \dim V_i^* = 0, \quad 0 \le i \le d$.

(3) $V_0, V_1, \cdots, V_d$: standard

$\Longrightarrow$

$V_d, V_{d-1}, \cdots, V_0$: standard

and no other standard orderings for the eigenspaces of A.

The same holds for the eigenspaces of A^*.

$(A, A^* ; \{V_i\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d})$ TD-system

standard ordering

3 more TD-systems

$(\{V_{d-i}\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d})$

$(\{V_i\}_{i=0}^{d}, \{V_{d-i}^*\}_{i=0}^{d})$

$(\{V_{d-i}\}_{i=0}^{d}, \{V_{d-i}^*\}_{i=0}^{d})$

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In what follows, we fix a TD-system

$$(A, A^* ; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d).$$

Set

$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d), \qquad 0 \le i \le d$$

<u>weight space</u>

$$U_0 = V_0^*, \qquad U_d = V_d$$

θ_i : eigenvalue of A on V_i
θ_i^* : eigenvalue of A^* on V_i^*

<u>Lemma</u>

$$(A - \theta_i) U_i \subseteq U_{i+1}, \qquad 0 \le i \le d, \qquad U_{d+1} = 0$$
$$(A^* - \theta_i^*) U_i \subseteq U_{i-1}, \qquad 0 \le i \le d, \qquad U_{-1} = 0$$

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<u>Proposition</u>

(1) $V = \bigoplus_{i=0}^{d} U_i$ <u>weight space decomposition</u>
(split decomposition)

(2) $U_0 + \cdots + U_i = V_0^* + \cdots + V_i^*, \qquad 0 \le i \le d$
$U_i + \cdots + U_d = V_i + \cdots + V_d, \qquad 0 \le i \le d$

(3) $\dim U_i = \dim V_i = \dim V_i^*, \qquad 0 \le i \le d$

(4) $\dim U_i = \dim U_{d-i}, \qquad 0 \le i \le d$

Proof

(1) $W = U_0 + U_1 + \cdots + U_d$ is $\langle A, A^* \rangle$-inv. by Lemma

$\supseteq U_0 = V_0^* \neq 0$

So $W = V$ by the irreducibility of V

$(U_0 + \cdots + U_i) \cap (U_{i+1} + \cdots + U_d)$

$\subseteq (V_0^* + \cdots + V_i^*) \cap (V_{i+1} + \cdots + V_d)$

$=: U_i'$

$W' = U_0' + U_1' + \cdots + U_{d-1}'$ is $\langle A, A^* \rangle$-inv.

$\subseteq V_0^* + \cdots + V_{d-1}^* \neq V$

$(A - \theta_{i+1}) U_i' \subseteq U_{i+1}'$, $(A - \theta_d) U_{d-1}' = 0$

$(A^* - \theta_i^*) U_i' \subseteq U_{i-1}'$, $(A^* - \theta_0^*) U_0' = 0$

So $W' = 0$

$(U_0 + \cdots + U_i) \cap (U_{i+1} + \cdots + U_d) = 0$

(2) $U_0 + \cdots + U_i \subseteq V_0^* + \cdots + V_i^*$

$V_0^* + \cdots + V_i^* = (A^* - \theta_{i+1}^*) \cdots (A^* - \theta_d^*) V$

$\subseteq U_0 + \cdots + U_i$ by Lemma + $V = \bigoplus_{j=0}^{d} U_j$

$U_i + \cdots + U_d \subseteq V_i + \cdots + V_d$

$V_i + \cdots + V_d = (A - \theta_{i-1}) \cdots (A - \theta_0) V$

$\subseteq U_i + \cdots + U_d$ by Lemma + $V = \bigoplus_{j=0}^{d} U_j$

(3) $U_i \simeq U_0 + \cdots + U_i / U_0 + \cdots + U_{i-1} = V_0^* + \cdots + V_i^* / V_0^* + \cdots + V_{i-1}^* \simeq V_i^*$

$U_i \simeq U_i + \cdots + U_d / U_{i+1} + \cdots + U_d = V_i + \cdots + V_d / V_{i+1} + \cdots + V_d \simeq V_i$

(4) $(A, A^*; \{V_{d-i}\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d})$ TD-system

$\tilde{U}_i = (V_0^* + \cdots + V_i^*) \cap (V_{d-i} + \cdots + V_0)$

Then by (3)

$$\begin{array}{ccccc} \dim \tilde{U}_i & = & \dim V_i^* & = & \dim V_{d-i} \\ & & \| & & \| \\ & & \dim U_i & & \dim U_{d-i} \end{array}$$

$\boxtimes$

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$V = \bigoplus_{i=0}^{d} U_i$ w. s. decomp.

$F_i : V \longrightarrow U_i$ projection

$R = A - \sum_{i=0}^{d} \theta_i F_i$ raising map

θ_i : eigenvalue of A on V_i

$L = A^* - \sum_{i=0}^{d} \theta_i^* F_i$ lowering map

θ_i^* : eigenvalue of A^* on V_i^*

$$\begin{cases} R U_i \subseteq U_{i+1}, & 0 \le i \le d, \quad U_{d+1} = 0 \\ L U_i \subseteq U_{i-1}, & 0 \le i \le d, \quad U_{-1} = 0 \end{cases}$$

Proof $R|_{U_i} = A - \theta_i |_{U_i}$

$L|_{U_i} = A^* - \theta_i^* |_{U_i}$

The result follows from Lemma. $\boxtimes$

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Proposition

(1) $R^{d-2i} : U_i \longrightarrow U_{d-i}$ bijection, $0 \le i \le \frac{d}{2}$

(2) $L^{d-2i} : U_{d-i} \longrightarrow U_i$ bijection, $0 \le i \le \frac{d}{2}$

Corollary

$\dim U_0 \le \dim U_1 \le \dim U_2 \le \cdots\cdots \ge \dim U_{d-1} \ge \dim U_d$

Theorem* (shape conjecture)

$\dim U_0 = 1.$

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<u>Proof of Proposition</u>

(1)

$$R^{d-2i}\big|_{U_i} = (A-\theta_{d-i-1})\cdots(A-\theta_{i+1})(A-\theta_i)\big|_{U_i}$$

$$\ker R^{d-2i}\big|_{U_i} = U_i \cap (V_{d-i-1}+\cdots+V_{i+1}+V_i)$$

$$\subseteq (V_0^*+\cdots+V_i^*) \cap (V_{d-i-1}+\cdots+V_0)$$

$$= 0$$

<u>claim</u>

<u>Proof of claim</u>

$$\left(A, A^*; \{V_{d-i}\}_{i=0}^d, \{V_i^*\}_{i=0}^d\right) \quad \text{TD-system}$$

reversed ordering

$$V = \bigoplus_{i=0}^d \tilde{U}_i \qquad \text{w.s. decomp.}$$

where $\tilde{U}_i = (V_0^*+\cdots+V_i^*) \cap (V_{d-i}+\cdots+V_0)$

Then
$$V_0^*+\cdots+V_i^* = \tilde{U}_0+\cdots+\tilde{U}_i$$
$$V_{d-i}+\cdots+V_0 = \tilde{U}_i+\cdots+\tilde{U}_d,$$

$$(V_0^*+\cdots+V_i^*) \cap (V_{d-i-1}+\cdots+V_0)$$
$$= (\tilde{U}_0+\cdots+\tilde{U}_i) \cap (\tilde{U}_{i+1}+\cdots+\tilde{V}_d) = 0 \quad //$$

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So $\ker R^{d-2i}\big|_{U_i} = 0$.

The mapping is injective.

Since $\dim U_i = \dim U_{d-i}$, the surjectivity follows from the injectivity.

(2)

$$L^{d-2i}\big|_{U_{d-i}} = (A^*-\theta_{i+1}^*)\cdots(A^*-\theta_{d-i}^*)\big|_{U_{d-i}}$$

$$\ker L^{d-2i}\big|_{U_{d-i}} = U_{d-i} \cap (V_{i+1}^*+\cdots+V_{d-i}^*)$$

$$\subseteq (V_{d-i}+\cdots+V_d) \cap (V_{i+1}^*+\cdots+V_d^*)$$

$$= 0$$

<u>claim</u>

$$\left(A, A^*; \{V_i\}_{i=0}^d, \{V_{d-i}^*\}_{i=0}^d\right) \quad \text{TD-system}$$

reversed ordering

$$\tilde{U}_i = (V_d^*+\cdots+V_{d-i}^*) \cap (V_i+\cdots+V_d)$$

$$V = \bigoplus_{i=0}^d \tilde{U}_i$$

$$V_d^*+\cdots+V_{d-i}^* = \tilde{U}_0+\cdots+\tilde{U}_i$$
$$V_i+\cdots+V_d = \tilde{U}_i+\cdots+\tilde{U}_d$$

$$(V_{d-i}+\cdots+V_d) \cap (V_{i+1}^*+\cdots+V_d^*)$$
$$= (\tilde{U}_{d-i}+\cdots+\tilde{U}_d) \cap (\tilde{U}_0+\cdots+\tilde{U}_{d-i-1}) = 0 \quad \blacksquare$$

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$A, A^* \in End(V)$ TD-pair
diameter d

$V = \bigoplus_{i=0}^{d} V_i$ eigenspace decomposition of A

$E_i : V \longrightarrow V_i$ projection

$A = \sum_{i=0}^{d} \theta_i E_i$, θ_i : eigenvalue of A on V_i

$$E_i = \prod_{\nu \neq i} \frac{A - \theta_\nu}{\theta_i - \theta_\nu}$$

$I = E_0 + E_1 + \cdots + E_d$, $E_i E_j = \delta_{ij} E_i$

$V = \bigoplus_{i=0}^{d} V_i^*$ eigenspace decomposition of A^*

$E_i^* : V \longrightarrow V_i^*$ projection

$A^* = \sum_{i=0}^{d} \theta_i^* E_i^*$, θ_i^* : eigenvalue of A^* on V_i^*

$$E_i^* = \prod_{\nu \neq i} \frac{A^* - \theta_\nu^*}{\theta_i^* - \theta_\nu^*}$$

$I = E_0^* + E_1^* + \cdots + E_d^*$, $E_i^* E_j^* = \delta_{ij} E_i^*$

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$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$, $0 \leq i \leq d$, $V_{-1}^* = V_{d+1}^* = 0$

$$E_j^* A E_i^* \begin{cases} = 0 & \text{if } |j-i| > 1 \\ \neq 0 & \text{if } |j-i| = 1 \end{cases}$$

$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}$, $0 \leq i \leq d$, $V_{-1} = V_{d+1} = 0$

$$E_j A^* E_i \begin{cases} = 0 & \text{if } |j-i| > 1 \\ \neq 0 & \text{if } |j-i| = 1 \end{cases}$$

<u>Proposition</u> $\ell = |d - 2i|$, $0 \leq i \leq d$

(1) $E_{d-i}^* A^\ell E_i^* : V_i^* \longrightarrow V_{d-i}^*$ bijection

(2) $E_{d-i} A^{*\ell} E_i : V_i \longrightarrow V_{d-i}$ bijection

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<u>Proof</u>

(1) <u>case</u> $d-i \geq i$

Set $W_i = V_0^* + \cdots + V_i^* = U_0 + \cdots + U_i$.

Then $V_i^* \simeq W_i / W_{i-1} \simeq U_i$

$$E_{i+1}^* A E_i^* : V_i^* \longrightarrow V_{i+1}^*$$

$$\simeq$$

$$A : W_i/W_{i-1} \longrightarrow W_{i+1}/W_i$$

$$\simeq$$

$$F_{i+1} A F_i : U_i \longrightarrow U_{i+1}$$

$$\|$$

$$F_{i+1} R F_i$$

Since $F_{d-i} A^\ell F_i : U_i \longrightarrow U_{d-i}$ is bijective,

so is $E_{d-i}^* A^\ell E_i^* : V_i^* \longrightarrow V_{d-i}^*$.

<u>case</u> $d-i < i$

Apply the same argument to the TD-system

$(A, A^* ; \{V_i\}_{i=0}^d , \{V_{d-i}^*\}_{i=0}^d)$

reversed ordering

(2) The same argument as in (1). ▨

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<u>Corollary</u>

(1) $E_j^* A^\ell E_i^* = 0$ if $|j-i| > \ell$

$\neq 0$ if $|j-i| = \ell$

(2) $E_j A^{*\ell} E_i = 0$ if $|j-i| > \ell$

$\neq 0$ if $|j-i| = \ell$

<u>Proof</u>

(1) <u>case</u> $\ell = j-i$, $d-i \geq j \geq i$

$$0 \neq E_{d-i}^* A^{d-2i} E_i^* = E_{d-i}^* A^{d-i-j} E_j^* A^\ell E_i^*$$

So $E_j^* A^\ell E_i^* \neq 0$.

Other cases are proved in the same way. ▨

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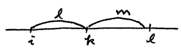
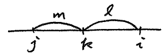
<u>Lemma</u>

(1) $$E_j^* A^m E_k^* A^\ell E_i^* = E_j^* A^{m+\ell} E_i^*$$

if $k = j - m = i + \ell$ or $k = j + m = i - \ell$.





(2) $$E_j A^{*m} E_k A^{*\ell} E_i = E_j A^{*m+\ell} E_i$$

if $k = j - m = i + \ell$ or $k = j + m = i - \ell$

<u>Proof</u>

(1) $$E_j^* A^{m+\ell} E_i^* = E_j^* A^m (E_0^* + E_1^* + \cdots + E_d^*) A^\ell E_i^*$$
$$= E_j^* A^m E_k^* A^\ell E_i^*$$

$\blacksquare$

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<u>4th lecture</u>

§4 TD-pairs : TD-relations and Terwilliger's Lemma

§4.1 TD-relations

<u>Theorem</u> (TD-relations)

$A, A^* \in End(V)$ TD-pair
diameter d

(1) $\exists \beta, \gamma, \delta \in \mathbb{C}$

(TD) $$A^3A^* - (\beta+1)(A^2A^*A - AA^*A^2) - A^*A^3 = \gamma(A^2A^* - A^*A^2) + \delta(AA^* - A^*A).$$

Moreover if $d \geq 3$, β, γ, δ are uniquely determined,
if $d = 2$, β is arbitrary, γ, δ are uniquely det.
if $d = 1$, β, γ are arbitrary, δ is uniq. det.

(2) $\exists \beta^*, \gamma^*, \delta^* \in \mathbb{C}$

(TD)* $$A^{*3}A - (\beta^*+1)(A^{*2}AA^* - A^*AA^{*2}) - AA^{*3} = \gamma^*(A^{*2}A - AA^{*2}) + \delta^*(A^*A - AA^*).$$

Moreover if $d \geq 3$, $\beta^*, \gamma^*, \delta^*$ are uniquely determined,
if $d = 2$, β^* is arbitrary, γ^*, δ^* are uniquely determined,
if $d = 1$, β^*, γ^* are arbitrary, δ^* is uniquely determined.

<u>Remark</u> $\beta = \beta^*$ holds.
(proof given later)

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<u>Proof</u> Notations as in §3.

(1) $\langle A \rangle \subseteq \mathrm{End}(V)$ subalg. generated by A

$\mathcal{L} = \mathrm{Span}\{XA^*Y - YA^*X \mid X, Y \in \langle A \rangle\} \subseteq \mathrm{End}(V)$
linear subspace

Then

$$\mathcal{L} = \mathrm{Span}\{E_jA^*E_i - E_iA^*E_j \mid 0 \le i, j \le d\}$$
$$= \mathrm{Span}\{E_{i+1}A^*E_i - E_iA^*E_{i+1} \mid 0 \le i \le d-1\}$$

by $E_jA^*E_i = 0$, $|j-i| > 1$.

In particular,

$\dim \mathcal{L} \le d$.

<u>claim</u> $\{A^iA^* - A^*A^i \mid 1 \le i \le d\}$ linearly independent

<u>proof of claim</u>

$$\sum_{i=0}^{r} c_i(A^iA^* - A^*A^i) = 0, \quad c_r \neq 0.$$

$$E_r^*(\qquad)E_0^* = \underbrace{c_r}_{\neq 0}\underbrace{(\theta_0^* - \theta_r^*)}_{\neq 0}\underbrace{E_r^*A^rE_0^*}_{\neq 0} \neq 0$$

※

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So $\dim \mathcal{L} = d$

and

$\{A^iA^* - A^*A^i \mid 1 \le i \le d\}$ is a basis of $\mathcal{L}$.

If $d \le 2$, the theorem holds.

Assume $d \ge 3$.

$$\mathcal{L} \ni A^2A^*A - AA^*A^2 = \sum_{i=1}^{r} c_i(A^iA^* - A^*A^i), \quad c_r \neq 0.$$

Enough to show $r = 3$.

Suppose $r \ge 4$.

$$E_r^*(A^2A^*A - AA^*A^2)E_0^* = 0 \quad \text{since } A^* = \sum_{i=0}^{d}\theta_i^*E_i^*$$

$$E_r^*\Big(\sum_{i=1}^{r} c_i(A^iA^* - A^*A^i)\Big)E_0^* = c_r(\theta_0^* - \theta_r^*)E_r^*A^rE_0^* \neq 0$$

※

Suppose $r \le 2$.

$$E_3^*(A^2A^*A - AA^*A^2)E_0^* = (\theta_1^* - \theta_2^*)E_3^*A^3E_0^* \neq 0$$

$$E_3^*\Big(\sum_{i=1}^{r} c_i(A^iA^* - A^*A^i)\Big)E_0^* = 0$$

※

(2) The same argument is available. ▨

§4.2 Eigenvalues

Theorem

$\left(A, A^* ; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d\right)$ TD-system

θ_i : eigenvalue of A on V_i, $0 \le i \le d$

θ_i^* : eigenvalue of A^* on V_i^*, $0 \le i \le d$

$\beta, \gamma, \delta, \beta^*, \gamma^*, \delta^*$: the parameters of the TD-relations

Then

(i)
$$\beta = \beta^* = \frac{\theta_{i+1} - \theta_i + \theta_{i-1} - \theta_{i-2}}{\theta_i - \theta_{i-1}} = \frac{\theta^*_{i+1} - \theta^*_i + \theta^*_{i-1} - \theta^*_{i-2}}{\theta^*_i - \theta^*_{i-1}}, \quad 2 \le i \le d-1.$$

(ii)
$$\gamma = \theta_{i+1} - \beta\theta_i + \theta_{i-1}, \quad 1 \le i \le d-1,$$
$$\gamma^* = \theta^*_{i+1} - \beta^*\theta^*_i + \theta^*_{i-1}, \quad 1 \le i \le d-1.$$

(iii)
$$\delta = \theta_{i+1}^2 - \beta\theta_{i+1}\theta_i + \theta_i^2 - \gamma(\theta_{i+1} + \theta_i), \quad 0 \le i \le d-1,$$
$$\delta^* = \theta^{*2}_{i+1} - \beta^*\theta^*_{i+1}\theta^*_i + \theta^{*2}_i - \gamma^*(\theta^*_{i+1} + \theta^*_i), \quad 0 \le i \le d-1.$$

Remark

(1) If $d \le 2$, β, β^* are arbitrary. So we set $\beta = \beta^*$.

(2) $\{x_i\}_{i\in\mathbb{Z}}$: β-sequence

if $(x_i - x_{i-1})\beta = x_{i+1} - x_i + x_{i-1} - x_{i-2}$, $i \in \mathbb{Z}$,

(β, γ)-sequence

if $\gamma = x_{i+1} - \beta x_i + x_{i-1}$, $i \in \mathbb{Z}$,

(β, γ, δ)-sequence

if $\delta = x_{i+1}^2 - \beta x_{i+1}x_i + x_i^2 - \gamma(x_{i+1} + x_i)$, $i \in \mathbb{Z}$.

β-seq. $\Longleftrightarrow$ (β, γ)-seq.

$$\begin{array}{rl} & \gamma = x_{i+1} - \beta x_i + x_{i-1} \\ -) & \gamma = x_i - \beta x_{i-1} + x_{i-2} \\ \hline & 0 = x_{i+1} - x_i - \beta(x_i - x_{i-1}) + x_{i-1} - x_{i-2} \end{array}$$

(β, γ)-seq. $\Longrightarrow$ (β, γ, δ)-seq.

$\Longleftarrow$ if $x_{i+1} \ne x_{i-1}$, $i \in \mathbb{Z}$

$$\begin{array}{rl} & \delta = x_{i+1}^2 - \beta x_{i+1}x_i + x_i^2 - \gamma(x_{i+1} + x_i) \\ -) & \delta = x_i^2 - \beta x_i x_{i-1} + x_{i-1}^2 - \gamma(x_i + x_{i-1}) \\ \hline & 0 = x_{i+1}^2 - x_{i-1}^2 - \beta x_i(x_{i+1} - x_{i-1}) - \gamma(x_{i+1} - x_{i-1}) \end{array}$$

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(3) $\{\theta_i\}_{i=0}^{d}$ is extended to $\{\theta_i\}_{i\in\mathbb{Z}}$ as a β-seq.

$\{\theta_i^*\}_{i=0}^{d}$ is extended to $\{\theta_i^*\}_{i\in\mathbb{Z}}$ as a β-seq.

(4) $\{x_i\}_{i\in\mathbb{Z}}$: β-sequence

$$\beta = q + q^{-1}$$

<u>type I</u> $q \neq \pm 1$

$$x_i = a + bq^i + cq^{-i}, \quad i \in \mathbb{Z}$$

<u>type II</u> $q = 1$

$$x_i = a + bi + ci^2, \quad i \in \mathbb{Z}$$

<u>type III</u> $q = -1$

$$x_i = a + b(-1)^i + c(-1)^i i, \quad i \in \mathbb{Z}$$

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<u>Proof of Theorem</u> Notations as in §3.

TD-relation

(TD) $A^3A^* - (\beta+1)(A^2A^*A - AA^*A^2) - A^*A^3 = \gamma(A^2A^* - A^*A^2) + \delta(AA^* - A^*A)$

$$\begin{cases} E_{i+1}(\text{LHS of TD})E_i = \left(\theta_{i+1}^3 - (\beta+1)(\theta_{i+1}^2\theta_i - \theta_{i+1}\theta_i^2) - \theta_i^3\right) E_{i+1}A^*E_i \\ E_{i+1}(\text{RHS of TD})E_i = \left(\gamma(\theta_{i+1}^2 - \theta_i^2) + \delta(\theta_{i+1} - \theta_i)\right) E_{i+1}A^*E_i \end{cases}$$

$$\theta_{i+1} - \theta_i \neq 0, \quad E_{i+1}A^*E_i \neq 0$$

So

$$\theta_{i+1}^2 - \beta\theta_{i+1}\theta_i + \theta_i^2 = \gamma(\theta_{i+1} + \theta_i) + \delta$$

$$\underset{(iii)}{(\beta, \gamma, \delta)\text{-seq.}} \Rightarrow \underset{(ii)}{(\beta, \gamma)\text{-seq.}} \Rightarrow \underset{(i)}{\beta\text{-seq.}}$$

$$\begin{cases} E_{i+3}^*(\text{LHS of TD})E_i^* = \left(\theta_i^* - (\beta+1)(\theta_{i+1}^* - \theta_{i+2}^*) - \theta_{i+3}^*\right) E_{i+3}^*A^3E_i^* \\ E_{i+3}^*(\text{RHS of TD})E_i^* = 0 \end{cases}$$

$$E_{i+3}^*A^3E_i^* = 0$$

So $\theta_i^* - (\beta+1)(\theta_{i+1}^* - \theta_{i+2}^*) - \theta_{i+3}^* = 0$

$\{\theta_i^*\}_{i\in\mathbb{Z}}$ is a β-sequence. $\beta = \beta^*$ ▨

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§ 4.3 TD-relations revisited

pre TD-pair: $A, A^* \in \mathrm{End}(V)$

$$V = \bigoplus_{i=0}^{d} U_i$$

$$F_i : V \longrightarrow U_i \quad \text{projection}, \quad 0 \le i \le d$$

$$R, L \in \mathrm{End}(V)$$

$$R U_i \subseteq U_{i+1}, \quad 0 \le i \le d, \qquad U_{d+1} = 0$$

$$L U_i \subseteq U_{i-1}, \quad 0 \le i \le d, \qquad U \;\; = 0$$

$$\theta_0, \theta_1, \cdots, \theta_d \in \mathbb{C}, \quad \theta_i \ne \theta_j$$

$$\theta_0^*, \theta_1^*, \cdots, \theta_d^* \in \mathbb{C}, \quad \theta_i^* \ne \theta_j^*$$

$$\begin{cases} A = R + \sum_{i=0}^{d} \theta_i F_i \\ A^* = L + \sum_{i=0}^{d} \theta_i^* F_i \end{cases} \qquad \underline{\text{pre TD-pair}}$$

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Then a pre TD-pair $A, A^* \in \mathrm{End}(V)$ are diagonalizable.

$$V = \bigoplus_{i=0}^{d} V_i \;:\; \text{eigenspace decomposition of } A$$

$$A|_{V_i} = \theta_i \;:\; \text{eigenvalue on } V_i$$

$$V = \bigoplus_{i=0}^{d} V_i^* \;:\; \text{eigenspace decomposition of } A^*$$

$$A^*|_{V_i^*} = \theta_i^* \;:\; \text{eigenvalue on } V_i^*$$

and

$$U_0 + \cdots + U_i = V_0^* + \cdots + V_i^*, \quad 0 \le i \le d$$

$$U_i + \cdots + U_d = V_i + \cdots + V_d, \quad 0 \le i \le d$$

$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d), \quad 0 \le i \le d$$

$$(A, A^*; \{V_i\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d}) \qquad \underline{\text{pre TD-system}}$$

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Theorem

$(A, A^* ; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ pre TD-system

$$A = R + \sum_{i=0}^d \theta_i F_i ,$$

$$A^* = L + \sum_{i=0}^d \theta_i^* F_i .$$

Assume $\{\theta_i\}_{i\in\mathbb{Z}}$ is a (β, γ, δ)-sequence,

$\{\theta_i^*\}_{i\in\mathbb{Z}}$ is a $(\beta, \gamma^*, \delta^*)$-sequence.

Set

$$\alpha_i = (\beta+1)\left(\theta_i\theta_i^* - \theta_{i+2}\theta_{i+2}^* + (\theta_{i+1}\theta_{i+2}^* + \theta_{i+2}\theta_{i+1}^*) - (\theta_i\theta_{i+1}^* + \theta_{i+1}\theta_i^*)\right)$$

Then

(1) (TD) $A^3A^* - (\beta+1)(A^2A^*A - AA^*A^2) - A^*A^3 = \gamma(A^2A^* - A^*A^2) + \delta(AA^* - A^*A)$

$\Longleftrightarrow$

$$R^3L - (\beta+1)(R^2LR - RLR^2) - LR^3 = \alpha_i R^2 \quad \text{on } U_i, \quad 0 \le i \le d-2$$

(2) $(\text{TD})^*$ $A^{*3}A - (\beta+1)(A^{*2}AA^* - A^*AA^{*2}) - AA^{*3} = \gamma^*(A^{*2}A - AA^{*2}) + \delta^*(A^*A - AA^*)$,

$\Longleftrightarrow$

$$L^3R - (\beta+1)(L^2RL - LRL^2) - LR^3 = -\alpha_i L^2 \quad \text{on } U_{i+2}, \quad 0 \le i \le d-2$$

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Proof

$$A = R + F, \quad F = \sum_{i=0}^d \theta_i F_i$$

$$A^* = L + F^*, \quad F^* = \sum_{i=0}^d \theta_i^* F_i$$

(1) $A^3A^* - (\beta+1)(A^2A^*A - AA^*A^2) - A^*A^3 = X_3 + X_2 + X_1 + X_0 + X_{-1}$

$$X_3 = R^3F^* - F^*R^3 - (\beta+1)(R^2F^*R - RF^*R^2)$$

$$\begin{aligned} X_2 = {} & R^3L - LR^3 + (R^2F + RFR + FR^2)F^* - F^*(R^2F + RFR + FR^2) \\ & - (\beta+1)(R^2LR - RLR^2) \\ & - (\beta+1)\{R^2F^*F + (RF+FR)F^*R - FF^*R^2 - RF^*(RF+FR)\} \end{aligned}$$

$$\begin{aligned} X_1 = {} & (R^2F + RFR + FR^2)L - L(R^2F + RFR + FR^2) \\ & + (RF^2 + FRF + F^2R)F^* - F^*(RF^2 + FRF + F^2R) \\ & - (\beta+1)\{R^2LF - FLR^2 + (RF+FR)LR - RL(RF+FR)\} \\ & - (\beta+1)\{(RF+FR)F^*F + F^2F^*R - FF^*(RF+FR) - RF^*F^2\} \end{aligned}$$

$$\begin{aligned} X_0 = {} & (RF^2 + FRF + F^2R)L - L(RF^2 + FRF + F^2R) + F^3F^* - F^*F^3 \\ & - (\beta+1)\{(RF+FR)LF + F^2LR - FL(RF+FR) - RLF^2\} \\ & - (\beta+1)(F^2F^*F - FF^*F^2) \end{aligned}$$

$$X_{-1} = F^3L - LF^3 - (\beta+1)(F^2LF - FLF^2)$$

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$$\gamma(A^2A^* - A^*A^2) + \delta(AA^* - A^*A) = Y_2 + Y_1 + Y_0 + Y_{-1}$$

$$Y_2 = \gamma(R^2F^* - F^*R^2)$$

$$Y_1 = \gamma(R^2L - LR^2) + \gamma\{(RF+FR)F^* - F^*(RF+FR)\} + \delta(RF^* - F^*R)$$

$$Y_0 = \gamma\{(RF+FR)L - L(RF+FR)\} + \gamma(F^2F^* - F^*F^2) + \delta(RL - LR) + \delta(FF^* - F^*F)$$

$$Y_{-1} = \gamma(F^2L - LF^2) + \delta(FL - LF)$$

on U_i

$X_3 = 0$ ok by $\{\theta_i^*\}_{i\in\mathbb{Z}}$ β-sequence

$X_2 = Y_2 \iff R^3L - (\beta+1)(R^2LR - RLR^2) - LR^3 = \alpha_i R^2$

$X_1 = Y_1$ ok by $\{\theta_i\}_{i\in\mathbb{Z}}$ $(\beta, \gamma\ \ )$-seq.

$X_0 = Y_0$ ok by $\{\theta_i\}_{i\in\mathbb{Z}}$ (β, γ, δ)-seq.

$X_{-1} = Y_{-1}$ ok by $\{\theta_i\}_{i\in\mathbb{Z}}$ (β, γ, δ)-seq.

(2) by the same argument. $\boxed{\diagup\!\!\diagup}$

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Theorem

$(A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ pre TD-system

$$A = R + \sum_{i=0}^d \theta_i F_i,$$

$$A^* = L + \sum_{i=0}^d \theta_i^* F_i.$$

Assume $\{\theta_i\}_{i\in\mathbb{Z}}$ is a (β, γ, δ)-sequence,

$\{\theta_i^*\}_{i\in\mathbb{Z}}$ is a $(\beta, \gamma^*, \delta^*)$-sequence.

Set $\beta = q + q^{-1}$.

Then

(1) (TD) $A^3A^* - (\beta+1)(A^2A^*A - AA^*A^2) - A^*A^3 = \gamma(A^2A^* - A^*A^2) + \delta(AA^* - A^*A)$

$\Longrightarrow$ $A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}$, $0 \le i \le d$,

$$V_{d+1} = 0, \quad V_{-1} = \begin{cases} V_d & \text{if } q^{d+1} = 1,\ q \neq \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

(2) $(\mathrm{TD})^*$ $A^{*3}A - (\beta+1)(A^{*2}AA^* - A^*AA^{*2}) - AA^{*3} = \gamma^*(A^{*2}A - AA^{*2}) + \delta^*(A^*A - AA^*)$

$\Longrightarrow$ $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$, $0 \le i \le d$,

$$V_{-1}^* = 0, \quad V_{d+1}^* = \begin{cases} V_0^* & \text{if } q^{d+1} = 1,\ q \neq \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

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Proof

(1) $\quad V_i \ni v$

$$\left(A^3A^* - (\beta+1)(A^2A^*A - AA^*A^2) - A^*A^3\right)v = \left(A^3 - (\beta+1)(\theta_i A^2 - \theta_i^2 A) - \theta_i^3\right)A^*v$$
$$= (A-\theta_i)\,(A^2 - \beta\theta_i A + \theta_i^2)\,A^*v$$

$$\left(\gamma(A^2A^* - A^*A^2) + \delta(AA^* - A^*A)\right)v = \left(\gamma(A^2 - \theta_i^2) + \delta(A-\theta_i)\right)A^*v$$
$$= (A-\theta_i)\left(\gamma(A+\theta_i) + \delta\right)A^*v$$

$-)$

$$0 = (A-\theta_i)\left(A^2 - (\beta\theta_i + \gamma)A + \theta_i^2 - \gamma\theta_i - \delta\right)A^*v$$

$$\beta\theta_i + \gamma = \beta\theta_i + (\theta_{i+1} - \beta\theta_i + \theta_{i-1}) = \theta_{i+1} + \theta_{i-1}$$

$$\theta_i^2 - \gamma\theta_i - \delta = \theta_i^2 - \gamma\theta_i - \left(\theta_{i+1}^2 - \beta\theta_{i+1}\theta_i + \theta_i^2 - \gamma(\theta_{i+1} + \theta_i)\right)$$
$$= \theta_{i+1}(\gamma - \theta_{i+1} + \beta\theta_i) = \theta_{i+1}\theta_{i-1}$$

So $\quad 0 = (A-\theta_{i+1})(A-\theta_i)(A-\theta_{i-1})A^*v$

$1 \le i \le d-1, \quad A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}$

$i = d, \quad A^*V_d \subseteq V_{d-1} + V_d \quad$ as $\quad A^* = L + \sum_{i=0}^{d} \theta_i^* F_i$

$i = 0, \quad A^*V_0 \subseteq V_{-1} + V_0 + V_1$

where $\quad V_{-1} = \{v \in V \mid Av = \theta_{-1} v\}$.

θ_{-1} is an eigenvalue of $A \iff q^{d+1} = 1, \quad q \ne \pm 1$
$\theta_{-1} = \theta_d$

$$\left.\begin{array}{ll} \text{Type I} & \theta_i = a + bq^i + cq^{-i} \\ \text{type II} & \theta_i = a + bi + ci^2 \\ \text{type III} & \theta_i = a + b(-1)^i + c(-1)^i i \end{array}\right\} \text{ and } \begin{array}{l}\theta_i \ne \theta_j \\ i \ne j \in \{0, 1, \cdots, d\}\end{array}$$ $\boxed{/\!/}$

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§4.4 Terwilliger's Lemma

$A, A^* \in End(V)$ diagonalizable

$V = \bigoplus_{i=0}^{d} V_i$ eigenspace decomposition of A

$A|_{V_i} = \theta_i$ eigenvalue of A on V_i

$V = \bigoplus_{i=0}^{d} V_i^*$ eigenspace decomposition of A^*

$A^*|_{V_i^*} = \theta_i^*$ eigenvalue of A^* on V_i^*

$E_i : V = \bigoplus_{\nu=0}^{d} V_\nu \longrightarrow V_i$ projection

$E_i^* : V = \bigoplus_{\nu=0}^{d} V_\nu^* \longrightarrow V_i^*$ projection

If $(A, A^*; \{V_i\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d})$ is a TD-system,

$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \le i \le d, \quad V_{-1}^* = V_{d+1}^* = 0$

$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}, \quad 0 \le i \le d, \quad V_{-1} = V_{d+1} = 0.$

If $(A, A^*; \{V_i\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d})$ is a pre TD-system,

$AV_i^* \subseteq V_0^* + \cdots + V_{i+1}^*, \quad 0 \le i \le d, \quad V_{d+1}^* = 0$

$A^*V_i \subseteq V_{i-1} + \cdots + V_d, \quad 0 \le i \le d, \quad V_{-1} = 0.$

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Terwilliger's Lemma

(1) $A(V_2^* + \cdots + V_d^*) \subseteq V_1^* + \cdots + V_d^*$

$\Longleftrightarrow$

$E_0^*(A - AE_0^*)(A^* - \theta_1^*) = 0$

(2) $A^*(V_0 + \cdots + V_{d-2}) \subseteq V_0 + \cdots + V_{d-1}$

$\Longleftrightarrow$

$E_d(A^* - A^*E_d)(A - \theta_{d-1}) = 0$

Proof

(1) $A(V_2^* + \cdots + V_d^*) \subseteq V_1^* + \cdots + V_d^*$

$\Longleftrightarrow$

$E_0^* A V_i^* = 0, \quad 2 \le i \le d$

$\Longleftrightarrow$

$E_0^* A E_i^* = 0, \quad 2 \le i \le d$

$\Longleftrightarrow$

$E_0^*(A - AE_0^*)(A^* - \theta_1^*) = 0$

$\blacksquare$

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§5 Classification of L-pairs

$(A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ L-system

$V = \bigoplus_{i=0}^d V_i, \quad \dim V_i = 1, \quad 0 \le i \le d$

eigenspace decomposition of A

θ_i : eigenvalue of A on V_i

$V = \bigoplus_{i=0}^d V_i^*, \quad \dim V_i^* = 1, \quad 0 \le i \le d$

eigenspace decomposition of A^*

θ_i^* : eigenvalue of A^* on V_i^*

(i) $A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \le i \le d, \quad V_{-1}^* = V_{d+1}^* = 0$

$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, \quad 0 \le i \le d, \quad V_{-1} = V_{d+1} = 0$

(ii) V is irreducible as an $\langle A, A^* \rangle$-module

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L - system

$\Longrightarrow$

pre L-system $(A, A^* ; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$

$\{\theta_i\}_{i=0}^d$: (β, γ, δ) - sequence

$\{\theta_i^*\}_{i=0}^d$: $(\beta, \gamma^*, \delta^*)$ - sequence

$+$

$$A(V_2^* + \cdots + V_d^*) \subseteq V_1^* + \cdots + V_d^*$$

$$A^*(V_0 + \cdots + V_{d-2}) \subseteq V_0 + \cdots + V_{d-1}$$

Such pre L-systems are classified by Terwilliger's Lemma.

Moreover such a pre L-system satisfies

(i) $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$, $0 \le i \le d$. $V_{-1}^* = V_{d+1}^* = 0$

$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}$, $0 \le i \le d$, $V_{-1} = V_{d+1} = 0$

by the converse theorem of TD-relations.

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§ 5.1 Transition matrices

$A, A^* \in End(V)$ pre L-pair

with data $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\lambda_i\}_{i=0}^{d-1})$

$$\begin{cases} \theta_i \neq \theta_j, & i \neq j \in \{0, 1, \cdots, d\} \\ \theta_i^* \neq \theta_j^*, & i \neq j \in \{0, 1, \cdots, d\} \\ \lambda_i \neq 0, & 0 \le i \le d-1 \end{cases}$$

Set $\lambda_{-1} = \lambda_d = 0$.

$V = \bigoplus_{i=0}^d U_i$, $\dim U_i = 1$, $0 \le i \le d$

$F_i : V \longrightarrow U_i$ projection

$R, L \in End(V)$

$$\begin{cases} RU_i = U_{i+1}, & 0 \le i \le d, \quad U_{d+1} = 0 \\ LU_i = U_{i-1}, & 0 \le i \le d, \quad U_{-1} = 0 \end{cases}$$

$$A = R + \sum_{i=0}^d \theta_i F_i$$

$$A^* = L + \sum_{i=0}^d \theta_i^* F_i$$

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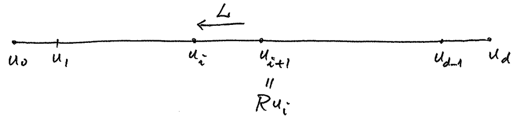
$$\lambda_i = \mathrm{tr}\, LR|_{U_i} = \mathrm{tr}\, RL|_{U_{i+1}}, \qquad -1 \le i \le d$$

$U_0 \ni u_0 \neq 0$

$$u_i = R^i u_0 \in U_i$$

$$L u_{i+1} = \lambda_i u_i, \qquad -1 \le i \le d$$



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Then A, A^* are diagonalizable.

$V = \bigoplus_{i=0}^{d} V_i$ eigenspace decomp. of A

$$A|_{V_i} = \theta_i$$

$E_i : V \longrightarrow V_i$ projection

$$A = \sum_{i=0}^{d} \theta_i E_i$$

$$E_i = \prod_{\nu \neq i} \frac{A - \theta_\nu}{\theta_i - \theta_\nu}$$

$V = \bigoplus_{i=0}^{d} V_i^*$ eigenspace decomp. of A^*

$$A^*|_{V_i^*} = \theta_i^*$$

$E_i^* : V \longrightarrow V_i^*$ projection

$$A^* = \sum_{i=0}^{d} \theta_i^* E_i^*$$

$$E_i^* = \prod_{\nu \neq i} \frac{A^* - \theta_\nu^*}{\theta_i^* - \theta_\nu^*}$$

$$\begin{cases} U_0 + U_1 + \cdots + U_i = V_0^* + V_1^* + \cdots + V_i^* \\ U_i + U_{i+1} + \cdots + U_d = V_i + V_{i+1} + \cdots + V_d \end{cases}$$

$$U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_i + V_{i+1} + \cdots + V_d)$$

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$U \ni u_0 \neq 0, \qquad u_i = R^i u_0 \in U_i$

$u_0, u_1, \cdots, u_d$: basis of V

$$U_i \simeq U_i + \cdots + U_d / U_{i+1} + \cdots + U_d$$
$$= V_i + \cdots + V_d / V_{i+1} + \cdots + V_d \simeq V_i$$

$\exists\, v_0, v_1, \cdots, v_d$: basis of V

$v_i \in V_i, \qquad F_i v_i = u_i$

$E_i u_i = v_i$

Lemma

(1) $(u_0, u_1, \cdots, u_d) = (v_0, v_1, \cdots, v_d)\, C$

$$C = (c_{ij}) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}$$

$$c_{ij} = \prod_{\nu=j}^{i-1} (\theta_i - \theta_\nu)^{-1}, \quad 0 \le j \le i \le d$$

(2) $(v_0, v_1, \cdots, v_d) = (u_0, u_1, \cdots, u_d)\, C^{-1}$

$$C^{-1} = (\tilde{c}_{ij}) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}$$

$$\tilde{c}_{ij} = \prod_{\nu=j+1}^{i} (\theta_j - \theta_\nu)^{-1}, \quad 0 \le j \le i \le d$$

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Proof

(1) $u_j = \sum_{i=j}^{d} c_{ij} v_i$

$$A u_j = \theta_j u_j + u_{j+1} = \sum_{i=j}^{d} (\theta_j c_{ij} + c_{i\,j+1}) v_i$$
$$A \sum_{i=j}^{d} c_{ij} v_i = \sum_{i=j}^{d} \theta_i c_{ij} v_i$$

$$\theta_j c_{ij} + c_{i\,j+1} = \theta_i c_{ij}$$

$$c_{ij} = (\theta_i - \theta_j)^{-1} c_{i\,j+1} = \cdots = (\theta_i - \theta_j)^{-1} \cdots (\theta_i - \theta_{i-1})^{-1} c_{ii}$$

(2) $v_j = \sum_{i=j}^{d} \tilde{c}_{ij} u_i$

$$A v_j = \theta_j v_j = \sum_{i=j}^{d} \theta_j \tilde{c}_{ij} u_i$$
$$A \sum_{i=j}^{d} \tilde{c}_{ij} u_i = \sum_{i=j}^{d} (\theta_i \tilde{c}_{ij} + \tilde{c}_{i-1\,j}) u_i$$

$$\theta_j \tilde{c}_{ij} = \theta_i \tilde{c}_{ij} + \tilde{c}_{i-1\,j}$$

$$\tilde{c}_{ij} = (\theta_j - \theta_i)^{-1} \tilde{c}_{i-1\,j} = \cdots = (\theta_j - \theta_i)^{-1} \cdots (\theta_j - \theta_{j+1})^{-1} \tilde{c}_{jj}$$

▨

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$$U_i \simeq U_0 + \cdots + U_i \,/\, U_0 + \cdots + U_{i-1}$$
$$= V_0^* + \cdots + V_i^* \,/\, V_0^* + \cdots + V_{i-1}^*$$
$$\simeq V_i^*$$

$\exists\ v_0^*, v_1^*, \cdots, v_d^*$: basis of V

$v_i^* \in V_i^*$, $\quad F_i v_i^* = u_i$

$E_i^* u_i = v_i^*$

<u>Lemma</u>

(1) $(u_0, u_1, \cdots, u_d) = (v_0^*, v_1^*, \cdots, v_d^*)\, C^*$

$$C^* = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = (c_{ij}^*)$$

$$c_{ij}^* = \prod_{\nu=i+1}^{j} \lambda_{\nu-1} (\theta_i^* - \theta_\nu^*)^{-1}, \qquad 0 \le i \le j \le d$$

(2) $(v_0^*, v_1^*, \cdots, v_d^*) = (u_0, u_1, \cdots, u_d)\, C^{*-1}$

$$C^{*-1} = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = (\tilde{c}_{ij}^*)$$

$$\tilde{c}_{ij}^* = \prod_{\nu=i}^{j-1} \lambda_\nu (\theta_j^* - \theta_\nu^*)^{-1}, \qquad 0 \le i \le j \le d$$

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§ 5.2 Equations derived from Terwilliger's Lemma

$(A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ pre L-system

$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\lambda_i\}_{i=0}^{d-1})$ data

<u>Proposition</u>

Set $\lambda_i' = \lambda_i - (\theta_i - \theta_d)(\theta_{i+1}^* - \theta_0^*)$, $\quad 0 \le i \le d-1$

$\lambda_{-1}' = \lambda_d' = 0$.

(1) $A(V_2^* + \cdots + V_d^*) \subseteq V_1^* + \cdots + V_d^*$

$\Longleftrightarrow$

$$\lambda_{i-1}' = \frac{\theta_i^* - \theta_1^*}{\theta_{i+1}^* - \theta_0^*} \lambda_i' + \lambda_0', \qquad 0 \le i \le d, \qquad \frac{\theta_d^* - \theta_1^*}{\theta_{d+1}^* - \theta_0^*} \lambda_d' \underset{\text{def}}{=} 0$$

(2) $A^*(V_0 + \cdots + V_{d-2}) \subseteq V_0 + \cdots + V_{d-1}$

$\Longleftrightarrow$

$$\lambda_i' = \frac{\theta_i - \theta_{d-1}}{\theta_{i-1} - \theta_d} \lambda_{i-1}' + \lambda_{d-1}', \qquad 0 \le i \le d, \qquad \frac{\theta_0 - \theta_{d-1}}{\theta_{-1} - \theta_d} \lambda_{-1}' \underset{\text{def}}{=} 0$$

<u>Proof</u>

(1) $A(V_2^* + \cdots + V_d^*) \subseteq V_1^* + \cdots + V_d^*$

$\Longleftrightarrow$

$E_0^*(A - AE_0^*)(A^* - \theta_1^*) = 0$ by Terwilliger's Lemma

$\Longleftrightarrow$

$E_0^*(A - a_0)(A^* - \theta_1^*) = 0$

$a_0 = \theta_0 + \dfrac{\lambda_0}{\theta_0^* - \theta_1^*}$

<u>proof</u> $E_0^* A E_0^* = a_0 \in \mathbb{C}$ $\quad V_0^* = U_0$

$E_0^* A E_0^* v_0^* = E_0^* A v_0^* \quad (v_0^* = u_0)$

$= E_0^*(\theta_0 u_0 + u_1)$

$= \theta_0 v_0^* + c_{01}^* v_0^*$

$= \left(\theta_0 + \dfrac{\lambda_0}{\theta_0^* - \theta_1^*}\right) v_0^*$

by Lemma on page 5-8

$V = \bigoplus_{i=0}^{d} U_i, \quad U_i = \mathbb{C} u_i, \quad u_i = R^i u_0$

$E_0^*(A - a_0)\underbrace{(A^* - \theta_1^*) u_i}_{= (\theta_i^* - \theta_1^*) u_i + \lambda_{i-1} u_{i-1}}$

$= E_0^*\Big((\theta_i^* - \theta_1^*) u_{i+1} + \big((\theta_i - a_0)(\theta_i^* - \theta_1^*) + \lambda_{i-1}\big) u_i + \lambda_{i-1}(\theta_{i-1} - a_0) u_{i-1}\Big) = 0$

So $(\theta_i^* - \theta_1^*) c_{0\,i+1}^* + \big((\theta_i - a_0)(\theta_i^* - \theta_1^*) + \lambda_{i-1}\big) c_{0i}^* + \lambda_{i-1}(\theta_{i-1} - a_0) c_{0\,i-1}^*$

$= \dfrac{\lambda_{i-1} \cdots \lambda_0}{(\theta_0^* - \theta_i^*) \cdots (\theta_0^* - \theta_1^*)} \left(\dfrac{\theta_i^* - \theta_1^*}{\theta_0^* - \theta_{i+1}^*} \lambda_i + \big((\theta_i - a_0)(\theta_i^* - \theta_1^*) + \lambda_{i-1}\big) + (\theta_{i-1} - a_0)(\theta_0^* - \theta_i^*) \right)$

$= 0, \quad 0 \le i \le d, \quad \dfrac{\theta_d^* - \theta_1^*}{\theta_0^* - \theta_{i+1}^*} \lambda_d = 0$

(2) $A^*(V_0 + \cdots + V_{d-2}) \subseteq V_0 + \cdots + V_{d-1}$

$\Longleftrightarrow$

$E_d(A^* - A^* E_d)(A - \theta_{d-1}) = 0$ by Terwilliger's Lemma

$\Longleftrightarrow$

$E_d(A^* - a_d^*)(A - \theta_{d-1}) = 0$

$a_d^* = \dfrac{\lambda_{d-1}}{\theta_d - \theta_{d-1}} + \theta_d^*$

<u>Proof</u> $E_d A^* E_d = a_d^* \in \mathbb{C}$

$E_d A^* E_d u_d = E_d A^* u_d \quad (V_d = U_d \ni \overset{v_d}{\underset{\|}{u_d}})$

$= E_d(\lambda_{d-1} u_{d-1} + \theta_d^* u_d)$

$= (\lambda_{d-1} c_{d\,d-1} + \theta_d^*) u_d$

$= \left(\dfrac{\lambda_{d-1}}{\theta_d - \theta_{d-1}} + \theta_d^*\right) u_d$

by Lemma on page 5-6

$E_d(A^* - a_d^*)(A - \theta_{d-1}) u_i$

$= E_d(A^* - a_d^*)\,(u_{i+1} + (\theta_i - \theta_{d-1}) u_i)$

$= E_d\Big((\theta_{i+1}^* - a_d^*) u_{i+1} + \big(\lambda_i + (\theta_i^* - a_d^*)(\theta_i - \theta_{d-1})\big) u_i + (\theta_i - \theta_{d-1}) \lambda_{i-1} u_{i-1}\Big)$

$= 0$

So $(\theta_{i+1}^* - a_d^*) c_{d\,i+1} + \big(\lambda_i + (\theta_i^* - a_d^*)(\theta_i - \theta_{d-1})\big) c_{di} + (\theta_i - \theta_{d-1}) \lambda_{i-1} c_{d\,i-1}$

$= \dfrac{1}{(\theta_d - \theta_i) \cdots (\theta_d - \theta_{d-1})} \left((\theta_{i+1}^* - a_d^*)(\theta_d - \theta_i) + \big(\lambda_i + (\theta_i^* - a_d^*)(\theta_i - \theta_{d-1})\big) + \dfrac{\theta_i - \theta_{d-1}}{\theta_d - \theta_{i-1}} \lambda_{i-1} \right)$

$= 0, \quad 0 \le i \le d, \quad \dfrac{\theta_0 - \theta_{d-1}}{\theta_d - \theta_{-1}} \lambda_{-1} \underset{\text{def}}{=} 0$

$\boxed{/\!/}$

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Assume

$\{\theta_i\}_{i=0}^{d}$: (β, γ, δ) – sequence

$\{\theta_i^*\}_{i=0}^{d}$: $(\beta, \gamma^*, \delta^*)$ – sequence

$\beta = q + q^{-1}$

type I ($q \neq \pm 1$)

$$\theta_i = a + bq^i + cq^{-i}, \quad i \in \mathbb{Z}$$

$$\theta_i^* = a^* + b^*q^i + c^*q^{-i}, \quad i \in \mathbb{Z}$$

Askey - Wilson parametrization,

$$\theta_i = \theta_0 + h q^{-i}(1-q^i)(1-sq^{i+1})$$

$$\theta_i^* = \theta_0^* + h^* q^{-i}(1-q^i)(1-s^*q^{i+1})$$

case $s \neq 0,\ s^* \neq 0$ q- Racah

type II ($q=1$) $q \to 1$, $(1-q)^2 h \longrightarrow h'$, $(1-q)^2 h^* \longrightarrow h^{*\prime}$

$\dfrac{1-s}{1-q} \longrightarrow s'$, $\dfrac{1-s^*}{1-q} \longrightarrow s^{*\prime}$

type III ($q=-1$) $q \to -1$, $(1+q) h \longrightarrow h'$, $(1+q) h^* \longrightarrow h^{*\prime}$

$\dfrac{1-s}{1+q} \longrightarrow s'$, $\dfrac{1-s^*}{1+q} \longrightarrow s^{*\prime}$

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type I

Lemma

$$\begin{cases} \lambda'_{i-1} = \dfrac{\theta_i^* - \theta_1^*}{\theta_{i+1}^* - \theta_0^*}\lambda'_i + \lambda'_0, & 0 \le i \le d-1 \\[2ex] \lambda'_i = \dfrac{\theta_i - \theta_{d-1}}{\theta_{i-1} - \theta_d}\lambda'_{i-1} + \lambda'_0, & 1 \le i \le d, \quad \lambda'_{d-1} = \lambda'_0 \end{cases}$$

$$\Longleftrightarrow \quad \lambda'_i = \frac{(1-q^{d-i})(1-q^{i+1})}{(1-q)(1-q^d)}\lambda'_0, \quad 0 \le i \le d-1$$

$$\Longleftrightarrow \quad \lambda_i = hh^* q^{-2i-1}(1-q^{i-d})(1-q^{i+1})(1-r_1q^{i+1})(1-r_2q^{i+1}), \quad 0 \le i \le d-1$$

$$r_1 r_2 = ss^* q^{d+1}$$

type II $\dfrac{1-r_1}{1-q} \longrightarrow r_1'$, $\dfrac{1-r_2}{1-q} \longrightarrow r_2'$

type III $\dfrac{1+r_1}{1+q} \longrightarrow r_1'$, $\dfrac{1+r_2 q^{d+1}}{1+q} \longrightarrow r_2' + d + 1$

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§5.3 Equations derived from TD-relations

$(A, A^* ; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ pre L-system

$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\lambda_i\}_{i=0}^{d-1})$ data

Assume

$\{\theta_i\}_{i=0}^d$: (β, γ, δ)-sequence

$\{\theta_i^*\}_{i=0}^d$: $(\beta, \gamma^*, \delta^*)$-sequence

Set

$$\alpha_i = (\beta+1)\left(\theta_i\theta_i^* - \theta_{i+2}\theta_{i+2}^* + (\theta_{i+1}\theta_{i+2}^* + \theta_{i+2}\theta_{i+1}^*) - (\theta_i\theta_{i+1}^* + \theta_{i+1}\theta_i^*)\right)$$

Proposition

(TD) $A^3A^* - (\beta+1)(A^2A^*A - AA^*A^2) - A^*A^3 = \gamma(A^2A^* - A^*A^2) + \delta(AA^* - A^*A)$

$\Longleftrightarrow$

$\lambda_{i-1} - (\beta+1)(\lambda_i - \lambda_{i+1}) - \lambda_{i+2} = \alpha_i, \quad 0 \le i \le d-2, \quad \lambda_{-1} = \lambda_d = 0$

$\Longleftrightarrow$

(TD)* $A^{*3}A - (\beta+1)(A^{*2}AA^* - A^*AA^{*2}) - AA^{*3} = \gamma^*(A^{*2}A - AA^{*2}) + \delta^*(A^*A - AA^*)$

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Proof

(TD) $\Longleftrightarrow$ $R^3L - (\beta+1)(R^2LR - RLR^2) - LR^3 = \alpha_i R^2$ on U_i, $0 \le i \le d-2$

$\Longleftrightarrow$ $\lambda_{i-1} - (\beta+1)(\lambda_i - \lambda_{i+1}) - \lambda_{i+2} = \alpha_i$, $0 \le i \le d-2$

(TD)* $\Longleftrightarrow$ $L^3R - (\beta+1)(L^2RL - LRL^2) - RL^3 = -\alpha_i L^2$ on U_{i+2}, $0 \le i \le d-2$

$\Longleftrightarrow$ $\lambda_{i+2}\lambda_{i+1}\lambda_i - (\beta+1)(\lambda_{i+1}\lambda_{i+1}\lambda_i - \lambda_{i+1}\lambda_i\lambda_i) - \lambda_{i+1}\lambda_i\lambda_{i-1}$
$= -\alpha_i\lambda_{i+1}\lambda_i$, $0 \le i \le d-2$

$\Longleftrightarrow$ $\lambda_{i+2} - (\beta+1)(\lambda_{i+1} - \lambda_i) - \lambda_{i-1} = -\alpha_i$, $0 \le i \le d-2$

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<u>Corollary</u>

$$A(V_2^* + \cdots + V_d^*) \subseteq V_1^* + \cdots + V_d^*$$

$$A^*(V_0 + \cdots + V_{d-2}) \subseteq V_0 + \cdots + V_{d-1}$$

$\Longrightarrow$

(TD) + (TD)*

In particular

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \le i \le d, \quad V_{-1}^* = V_{d+1}^* = 0$$

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, \quad 0 \le i \le d, \quad V_{-1} = V_{d+1} = 0.$$

<u>Proof</u>

Use the Proposition on page 5–9, the Lemma on page 5–13 and the Theorem on page 4–13. $\blacksquare$

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§5.4 The irreducibility

L-system $(A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$

$\Longrightarrow$

pre L-system $(A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$

data $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\lambda_i\}_{i=0}^{d-1})$

$\{\theta_i\}_{i\in\mathbb{Z}}$: (β, γ, δ)-sequence

$\{\theta_i^*\}_{i\in\mathbb{Z}}$: $(\beta, \gamma^*, \delta^*)$-sequence

$\beta = q + q^{-1}$

type I $(q \neq \pm 1)$

$$\lambda_i = h h^* q^{-2i-1}(1-q^{i+1})(1-q^{i-d})(1-r_1 q^{i+1})(1-r_2 q^{i+1}), \quad 0 \le i \le d-1$$

$$r_1 r_2 = s s^* q^{d+1}$$

type II, III limiting cases

Call such pre L-system <u>special</u>.

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$(A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ special pre L-system

$\Longrightarrow$

$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \le i \le d, \quad V_{-1}^* = V_{d+1}^* = 0$

$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}, \quad 0 \le i \le d, \quad V_{-1} = V_{d+1} = 0$

by Prop. on page 5-9
Lemma on page 5-13
Cor on page 5-16

So L-system

$\Longleftrightarrow$

special pre L-system

+
the irreducibility

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$(A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ special pre L-system

E_i : $V = \bigoplus_{\nu=0}^{d} V_\nu \longrightarrow V_i$ projection

E_i^* : $V = \bigoplus_{\nu=0}^{d} V_\nu^* \longrightarrow V_i^*$ projection

<u>Proposition</u>

V : irreducible as an $\langle A, A^* \rangle$-module

$\Longleftrightarrow$

$E_0^* V_0 \neq 0$

<u>Proof</u>

($\Longrightarrow$) Suppose $E_0^* V_0 = 0$.

Then $V_0 \subseteq V_1^* + \cdots + V_d^*$.

Set $W_i = (V_0 + \cdots + V_i) \cap (V_{i+1}^* + \cdots + V_d^*)$

$W = W_0 + W_1 + \cdots + W_{d-1}$.

Then $(A - \theta_i) W_i \subseteq W_{i-1}, \quad (A - \theta_0) W_0 = 0,$

$(A^* - \theta_{i+1}^*) W_i \subseteq W_{i+1}, \quad (A^* - \theta_d^*) W_{d-1} = 0,$

and so W is invariant under A, A^*.

$0 \neq V_0 = W_0 \subseteq W \subseteq V_1^* + \cdots + V_d^* \neq V$.

V is not irreducible. #

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$(\Leftarrow)$

$V \supseteq W \neq 0 \quad : \quad \langle A, A^* \rangle$ – submodule

$$W = \bigoplus_{i=0}^{d} W \cap V_i$$

$$i_0 = \mathrm{Min}\{ i \mid W \cap V_i \neq 0 \}$$

<u>claim</u> $\quad i_0 = 0$

$$V = \bigoplus_{i=0}^{d} U_i$$

$$U_i + \cdots + U_d = V_i + \cdots + V_d$$

$$\begin{aligned} U_{i_0} &\simeq U_{i_0} + \cdots + U_d / U_{i_0+1} + \cdots + U_d \\ &= V_{i_0} + \cdots + V_d / V_{i_0+1} + \cdots + V_d \\ &\simeq V_{i_0} \end{aligned}$$

$$\begin{aligned} (A^* - \theta^*_{i_0}) V_{i_0} &= (A^* - \theta^*_{i_0}) U_{i_0} \quad \mathrm{mod}\ U_{i_0+1} + \cdots + U_d \\ &= L U_{i_0} \\ &= U_{i_0 - 1} \end{aligned}$$

$i_0 = 0$ by the minimality. //

Then

$W \supseteq V_0 \quad$ as $\dim V_0 = 1$.

$W \supseteq E^*_0 V_0 \neq 0 \quad$ as $\dim V^*_0 = 1$

$\quad\quad \| $

$\quad V^*_0 = U_0$

$W \supseteq U_i = (A - \theta_{i-1}) \cdots (A - \theta_0) U_0 \quad \square$

$W = V$

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<u>Lemma</u>

$$E^*_0 V_0 \neq 0$$

$$\Longleftrightarrow$$

$$\sum_{i=0}^{d} \frac{\lambda_0 \lambda_1 \cdots\cdots \lambda_{i-1}}{(\theta_0 - \theta_1) \cdots (\theta_0 - \theta_i)(\theta^*_0 - \theta^*_1) \cdots (\theta^*_0 - \theta^*_i)} \neq 0$$

<u>Proof</u>

$$V_0 \ni v_0 = \sum_{i=0}^{d} \tilde{c}_{i0} u_i$$

$$\tilde{c}_{i0} = \frac{1}{(\theta_0 - \theta_1) \cdots (\theta_0 - \theta_i)} \quad \text{by Lemma on page 5–6}$$

$$U_i \ni u_i = c^*_{0i} v^*_0 + c^*_{1i} v^*_1 + \cdots$$

$$c^*_{0i} = \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{(\theta^*_0 - \theta^*_1)(\theta^*_0 - \theta^*_2) \cdots (\theta^*_0 - \theta^*_i)} \quad \text{by Lemma on page 5–8}$$

$$v_0 = \left(\sum_{i=0}^{d} \tilde{c}_{i0} c^*_{0i} \right) v^*_0 + \cdots$$

$\square$

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<u>q-Racah case</u>   type I + $s \neq 0,\ s^* \neq 0$

Set

$$\hat{\lambda}_i = \frac{hh^*}{s^*} q^{-2i-1} (1-q^{i+1})(1-q^{i-d})(r_1 - s^* q^{i+1})(r_2 - s^* q^{i+1}),$$

$$0 \leq i \leq d-1$$

<u>Theorem</u>

$$\sum_{i=0}^{d} \frac{\lambda_0 \cdots\cdots \lambda_{i-1}}{(\theta_0-\theta_1)\cdots(\theta_0-\theta_i)(\theta_0^*-\theta_1^*)\cdots(\theta_0^*-\theta_i^*)}$$

$$= \frac{\hat{\lambda}_0 \cdots \hat{\lambda}_{d-1}}{(\theta_0-\theta_1)\cdots(\theta_0-\theta_d)(\theta_0^*-\theta_1^*)\cdots(\theta_0^*-\theta_d^*)}$$

<u>Cor</u>

V : irreducible as an $\langle A, A^* \rangle$-module

$\Longleftrightarrow$

$\hat{\lambda}_i \neq 0 \ , \quad 0 \leq i \leq d-1.$

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<u>Proof of Theorem</u>   out line

$$LHS = {}_3\phi_2 \left(\begin{matrix} r_1 q,\ r_2 q,\ q^{-d} \\ sq^2,\ s^* q^2 \end{matrix} ; q, q \right)$$

calculation

$$= RHS \qquad \boxed{}$$

q- analogue of Pfaff-Saalschütz formula.

<u>Pfaff-Saalschütz formula</u>

$${}_3F_2 \left(\begin{matrix} a,\ b,\ -n \\ c,\ 1+a+b-c-n \end{matrix} ; 1 \right) = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n},$$

$$n = 0, 1, 2, \cdots$$

<u>q-analogue</u>

$${}_3\phi_2 \left(\begin{matrix} a,\ b,\ q^{-n} \\ c,\ abc^{-1} q^{1-n} \end{matrix} ; q, q \right) = \frac{(\frac{c}{a};q)_n (\frac{c}{b};q)_n}{(c;q)_n (\frac{c}{ab};q)_n},$$

$$n = 0, 1, 2, \cdots$$

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$A, A^* \in End(V)$ TD-pair

$\Longrightarrow$

4 TD-systems | data

$\left(\{V_i\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d}\right)$ $\left(\{\theta_i\}_{i=0}^{d}, \{\theta_i^*\}_{i=0}^{d}, \{\lambda_i\}_{i=0}^{d-1}\right)$

$\left(\{V_{d-i}\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d}\right)$ $\left(\{\theta_{d-i}\}_{i=0}^{d}, \{\theta_i^*\}_{i=0}^{d}, \{\hat{\lambda}_i\}_{i=0}^{d-1}\right)$

$\left(\{V_i\}_{i=0}^{d}, \{V_{d-i}^*\}_{i=0}^{d}\right)$ $\left(\{\theta_i\}_{i=0}^{d}, \{\theta_{d-i}^*\}_{i=0}^{d}, \{\hat{\lambda}_{d-1-i}\}_{i=0}^{d-1}\right)$

$\left(\{V_{d-i}\}_{i=0}^{d}, \{V_{d-i}^*\}_{i=0}^{d}\right)$ $\left(\{\theta_{d-i}\}_{i=0}^{d}, \{\theta_{d-i}^*\}_{i=0}^{d}, \{\lambda_{d-1-i}\}_{i=0}^{d-1}\right)$

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Proof

$$\lambda_{i-1} - (\beta+1)(\lambda_i - \lambda_{i+1}) - \lambda_{i+2} = \alpha_i, \quad 0 \le i \le d-2, \ \lambda_{-1} = \lambda_d = 0$$

$$\alpha_i = (\beta+1)\left(\theta_i\theta_i^* - \theta_{i+2}\theta_{i+2}^* + (\theta_{i+1}\theta_{i+2}^* + \theta_{i+2}\theta_{i+1}^*) - (\theta_i\theta_{i+1}^* + \theta_{i+1}\theta_i^*)\right)$$

$$\left(A, A^* ; \{V_{d-i}\}_{i=0}^{d}, \{V_{d-i}^*\}_{i=0}^{d}\right)$$

data $\left(\{\underbrace{\theta_{d-i}}_{=\tilde{\theta}_i}\}_{i=0}^{d}, \{\underbrace{\theta_{d-i}^*}_{=\tilde{\theta}_i^*}\}_{i=0}^{d}, \{\tilde{\lambda}_i\}_{i=0}^{d-1}\right)$

$$\tilde{\alpha}_i = (\beta+1)\left(\tilde{\theta}_i\tilde{\theta}_i^* - \tilde{\theta}_{i+2}\tilde{\theta}_{i+2}^* + (\tilde{\theta}_{i+1}\tilde{\theta}_{i+2}^* + \tilde{\theta}_{i+2}\tilde{\theta}_{i+1}^*) - (\tilde{\theta}_i\tilde{\theta}_{i+1}^* + \tilde{\theta}_{i+1}\tilde{\theta}_i^*)\right)$$

$$= -\alpha_{d-i-2}$$

$$= \underbrace{\lambda_{d-i}}_{=\tilde{\lambda}_{i-1}} - (\beta+1)(\underbrace{\lambda_{d-i-1}}_{=\tilde{\lambda}_i} - \underbrace{\lambda_{d-i-2}}_{=\tilde{\lambda}_{i+1}}) - \underbrace{\lambda_{d-i-3}}_{=\tilde{\lambda}_{i+2}} \qquad \blacksquare$$

$$\left(A^*, A ; \{V_i^*\}_{i=0}^{d}, \{V_i\}_{i=0}^{d}\right)$$

data $\left(\{\theta_i^*\}_{i=0}^{d}, \{\theta_i\}_{i=0}^{d}, \{\lambda_i\}_{i=0}^{d}\right)$

Proof α_i is symmetric w.r.t. $\{\theta_i\}_{i=0}^{d} \longleftrightarrow \{\theta_i^*\}_{i=0}^{d}$

$\blacksquare$

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<u>6th lecture</u>

§6 Classification of dual systems of orth. polys.

§6.1 Standard basis and the associated tridiagonal matrices

$(A, A^* ; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ L-system

$V_i \ni w_i \neq 0, \quad 0 \le i \le d$

$w_0, w_1, \cdots, w_d$ basis of V

the matrices of A, A^*

$$D = \begin{pmatrix} \theta_0 & & 0 \\ & \theta_1 & \\ 0 & & \ddots \; \theta_d \end{pmatrix}$$

$$B^* = \begin{pmatrix} a_0^* & b_0^* & & & 0 \\ c_1^* & a_1^* & b_1^* & & \\ & \ddots & \ddots & \ddots & \\ & & c_{d-1}^* & a_{d-1}^* & b_{d-1}^* \\ 0 & & & c_d^* & a_d^* \end{pmatrix} \in \mathcal{M}$$

the class of matrices that are
<u>tridiagonal</u>
<u>irreducible</u>
<u>diagonalizable</u>

$w_0, w_1, \cdots, w_d$ <u>standard</u> if $w_i \in V_i, \quad 0 \le i \le d$,

and

$B^* \in \mathcal{M}(\theta_0^*)$ the subclass of $\mathcal{M}$ that has <u>row sum θ_0^*</u>

$c_i^* + a_i^* + b_i^* = \theta_0^*, \quad 1 \le i \le d-1$

$a_0^* + b_0^* = c_d^* + a_d^* = \theta_0^*$

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$V_i^* \ni w_i \neq 0, \quad 0 \le i \le d$

$w_0, w_1, \cdots, w_d$ basis of V

the matrices of A, A^*

$$B = \begin{pmatrix} a_0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{d-1} & a_{d-1} & b_{d-1} \\ 0 & & & c_d & a_d \end{pmatrix} \in \mathcal{M}$$

tridiagonal
irreducible
diagonalizable

$$D^* = \begin{pmatrix} \theta_0^* & & 0 \\ & \theta_1^* & \\ 0 & & \ddots \; \theta_d^* \end{pmatrix}$$

$w_0, w_1, \cdots, w_d$ <u>dual</u> <u>standard</u> if $w_i \in V_i^*, \quad 0 \le i \le d$,

and

$B \in \mathcal{M}(\theta_0)$ the subclass of $\mathcal{M}$ that has <u>row sum θ_0</u>

$c_i + a_i + b_i = \theta_0, \quad 1 \le i \le d$

$a_0 + b_0 = c_d + a_d = \theta_0$

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Proposition

(1) $V_0^* \ni v_0^* \neq 0$, $\quad w_i = E_i v_0^* \in V_i$, $\quad 0 \le i \le d$

projection of v_0^* onto V_i

Then

$w_0, w_1, \cdots, w_d$ are a standard basis.

If $w_0', w_1', \cdots, w_d'$ are a standard basis

then $\exists\, \xi \in \mathbb{C}$, $\quad w_i' = \xi w_i$, $\quad 0 \le i \le d$.

(2) $V_0 \ni v_0 \neq 0$, $\quad w_i = E_i^* v_0 \in V_i^*$, $\quad 0 \le i \le d$

projection of v_0 onto V_i^*

Then

$w_0, w_1, \cdots, w_d$ are a dual standard basis.

If $w_0', w_1', \cdots, w_d'$ are a dual standard basis,

then $\exists\, \xi \in \mathbb{C}$, $\quad w_i' = \xi w_i$, $\quad 0 \le i \le d$.

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Proof

(1) $V = \bigoplus_{i=0}^{d} U_i$ weight space decomposition

$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d)$$

$U_0 = V_0^* \ni v_0^* = u_0$, $\qquad R^i u_0 = u_i \in U_i$

$$E_i = \prod_{\nu \neq i} \frac{A - \theta_\nu}{\theta_i - \theta_\nu} = \prod_{\nu=i+1}^{d} \frac{A - \theta_\nu}{\theta_i - \theta_\nu} \cdot \prod_{\nu=0}^{i-1} \frac{A - \theta_\nu}{\theta_i - \theta_\nu}$$

$$w_i = E_i u_0 = \left(\prod_{\nu=0}^{i-1} \frac{1}{\theta_i - \theta_\nu} \right) \underbrace{\prod_{\nu=i+1}^{d} \frac{A - \theta_\nu}{\theta_i - \theta_\nu} u_i}_{\substack{\parallel \\ u_i \ \bmod\ U_{i+1} + \cdots + U_d}}$$

$$= \frac{1}{(\theta_i - \theta_0) \cdots (\theta_i - \theta_{i-1})} u_i \quad \bmod\ U_{i+1} + \cdots + U_d$$

$$= V_{i+1} + \cdots + V_d$$

$\neq 0$.

So $0 \neq w_i \in V_i$

and $\quad w_0 + w_1 + \cdots + w_d = v_0^* \in V_0^*$

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$$A^*(w_0, w_1, \cdots, w_d) = (w_0, w_1, \cdots, w_d) B^*$$

$$A^* w_i = b_{i-1}^* w_{i-1} + a_i^* w_i + c_{i+1}^* w_{i+1}$$

$$A^* v_0^* = \theta_0^* v_0^* = \theta_0^* (w_0 + w_1 + \cdots + w_d)$$

$$\| \qquad \sum_{i=0}^{d} A^* w_i$$

So $\quad b_i^* + a_i^* + c_i^* = \theta_0^* \quad$ the coeff. of w_i.

Suppose $w_0', w_1', \cdots, w_d'$ is a standard basis.

Then $\exists\, \xi_i \in \mathbb{C}$, $\quad w_i' = \xi_i w_i$,

and the matrix of A^* w.r.t. $w_0', w_1', \cdots, w_d'$ is

$$B^{*\prime} = \begin{pmatrix} a_0^* & b_0^* \frac{\xi_1}{\xi_0} & & 0 \\ c_1^* \frac{\xi_0}{\xi_1} & a_1^* & b_1^* \frac{\xi_2}{\xi_1} & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix}$$

ith row of $B^{*\prime} = 0, \cdots, 0, \ c_i^* \frac{\xi_{i-1}}{\xi_i}, \ a_i^*, \ b_i^* \frac{\xi_{i+1}}{\xi_i}, 0, \cdots, 0$

So $\quad c_i^* \dfrac{\xi_{i-1}}{\xi_i} + a_i^* + b_i^* \dfrac{\xi_{i+1}}{\xi_i} = \theta_0^* = c_i^* + a_i^* + b_i^*$.

Inductively $\xi_0 = 1, \xi_1 = 1, \xi_2 = 1, \cdots, \xi_{i+1} = 1, \cdots, \xi_d = 1$ $\square$

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$(A, A^* ; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ $\quad$ L-system

data $\left(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\lambda_i\}_{i=0}^{d-1} \right)$

$(A, A^* ; \{V_{d-i}\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ $\quad$ L-system

↖ reversed

data $\left(\{\theta_{d-i}\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\hat{\lambda}_i\}_{i=0}^d \right)$

dual standard basis $\quad w_0, w_1, \cdots, w_d$

$$A(w_0, w_1, \cdots, w_d) = (w_0, w_1, \cdots, w_d) B$$

$$B = \begin{pmatrix} a_0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{d-1} & a_{d-1} & b_{d-1} \\ 0 & & & c_d & a_d \end{pmatrix} \in \mathcal{M}(\theta_0)$$

standard basis $\quad w_0, w_1, \cdots, w_d$

$$A^*(w_0, w_1, \cdots, w_d) = (w_0, w_1, \cdots, w_d) B^*$$

$$B^* = \begin{pmatrix} a_0^* & b_0^* & & & 0 \\ c_1^* & a_1^* & b_1^* & & \\ & \ddots & \ddots & \ddots & \\ & & c_{d-1}^* & a_{d-1}^* & b_{d-1}^* \\ 0 & & & c_d^* & a_d^* \end{pmatrix} \in \mathcal{M}(\theta_0^*)$$

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Theorem

(1) $c_i = \hat{\lambda}_{i-1} \dfrac{(\theta^*_i - \theta^*_{i+1}) \cdots (\theta^*_i - \theta^*_d)}{(\theta^*_{i-1} - \theta^*_i) \cdots (\theta^*_{i-1} - \theta^*_d)}, \quad 1 \le i \le d$

(2) $b_i = \lambda_i \dfrac{(\theta^*_i - \theta^*_{i-1}) \cdots (\theta^*_i - \theta^*_0)}{(\theta^*_{i+1} - \theta^*_i) \cdots (\theta^*_{i+1} - \theta^*_0)}, \quad 0 \le i \le d-1$

(3) $c^*_i = \hat{\lambda}_{d-i} \dfrac{(\theta_i - \theta_{i+1}) \cdots (\theta_i - \theta_d)}{(\theta_{i-1} - \theta_i) \cdots (\theta_{i-1} - \theta_d)}, \quad 1 \le i \le d$

(4) $b^*_i = \lambda_i \dfrac{(\theta_i - \theta_{i-1}) \cdots (\theta_i - \theta_0)}{(\theta_{i+1} - \theta_i) \cdots (\theta_{i+1} - \theta_0)}, \quad 0 \le i \le d-1$

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Proof

(1) $(A, A^* ; \{V_{d-i}\}_{i=0}^d, \{V^*_i\}_{i=0}^d)$ L-system

data $(\{\theta_{d-i}\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d, \{\hat{\lambda}_i\}_{i=0}^{d-1})$

$V = \bigoplus_{i=0}^d \hat{U}_i$ w. s. decomp.

$\hat{U}_i = (V^*_0 + \cdots + V^*_i) \cap (V_{d-i} + \cdots + V_0)$

$\hat{F}_i : V \longrightarrow \hat{U}_i$ projection

$A = \hat{R} + \sum_{i=0}^d \theta_{d-i} \hat{F}_i, \quad \hat{R}\hat{U}_i = \hat{U}_{i+1},$

$A^* = \hat{L} + \sum_{i=0}^d \theta^*_i \hat{F}_i, \quad \hat{L}\hat{U}_i = \hat{U}_{i-1}$

dual standard basis $w_0, w_1, \cdots, w_d$

$V_0 = \hat{U}_d \ni \hat{u}_d \neq 0, \quad \hat{u}_i = \hat{R}^i \hat{u}_0, \quad 0 \neq \hat{u}_0 \in \hat{U}_0$

$$w_i = E^*_i \hat{u}_d = \left(\prod_{\nu=0}^{i-1} \frac{A^* - \theta^*_\nu}{\theta^*_i - \theta^*_\nu}\right) \underbrace{\left(\prod_{\nu=i+1}^{d} \frac{A^* - \theta^*_\nu}{\theta^*_i - \theta^*_\nu}\right) \hat{u}_d}_{= \frac{\hat{\lambda}_i \cdots \hat{\lambda}_{d-1}}{(\theta^*_i - \theta^*_{i+1}) \cdots (\theta^*_i - \theta^*_d)} \hat{u}_i}$$

$$= \frac{\hat{\lambda}_i \cdots \hat{\lambda}_{d-1}}{(\theta^*_i - \theta^*_{i+1}) \cdots (\theta^*_i - \theta^*_d)} \hat{u}_i \quad \text{mod} \quad \underbrace{\hat{U}_0 + \cdots + \hat{U}_{i-1}}_{= V^*_0 + \cdots + V^*_{i-1}}$$

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$$A w_i = b_{i-1} w_{i-1} + a_i w_i + c_{i+1} w_{i+1}$$

mod $\hat{U}_0 + \cdots + \hat{U}_i$

$$\text{RHS} = c_{i+1} \frac{\hat{\lambda}_{i+1} \cdots \hat{\lambda}_{d-1}}{(\theta^*_{i+1} - \theta^*_{i+2}) \cdots (\theta^*_{i+1} - \theta^*_d)} \hat{u}_{i+1}$$

LHS

$$A = \hat{R} + \sum_{i=0}^{d} \theta_{d-i} \hat{F}_i$$

$$A \hat{u}_i = \hat{u}_{i+1} \quad \text{mod } \hat{U}_i$$

$$A(\hat{U}_0 + \cdots + \hat{U}_{i-1}) \subseteq \hat{U}_0 + \cdots + \hat{U}_i$$

$$= \frac{\hat{\lambda}_i \cdots \hat{\lambda}_{d-1}}{(\theta^*_i - \theta^*_{i+1}) \cdots (\theta^*_i - \theta^*_d)} \hat{u}_{i+1}$$

So

$$c_{i+1} = \hat{\lambda}_i \frac{(\theta^*_{i+1} - \theta^*_{i+2}) \cdots (\theta^*_{i+1} - \theta^*_d)}{(\theta^*_i - \theta^*_{i+1}) \cdots (\theta^*_i - \theta^*_d)}, \qquad 0 \le i \le d-1$$

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(2) Reverse the ordering $V^*_0, V^*_1, \ldots, V^*_d$

$$(A, A^*; \{V_i\}_{i=0}^d, \{V^*_{d-i}\}_{i=0}^d)$$

standard basis $w_0, w_1, \ldots, w_d$, $w_i \in V^*_i$

$$A(w_d, \ldots, w_1, w_0) = (w_d, \ldots, w_1, w_0) \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} B \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} B \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} = \begin{pmatrix} a_d & c_d & & & 0 \\ b_{d-1} & a_{d-1} & c_{d-1} & & \\ & \ddots & \ddots & \ddots & \\ & & b_1 & a_1 & c_1 \\ 0 & & & b_0 & a_0 \end{pmatrix}$$

$$= \begin{pmatrix} a'_0 & b'_0 & & & 0 \\ c'_1 & a'_1 & b'_1 & & \\ & \ddots & \ddots & \ddots & \\ & & c'_{d-1} & a'_{d-1} & b'_{d-1} \\ 0 & & & c'_d & a'_d \end{pmatrix}$$

$$b_{d-i} = c'_i \underset{\text{by (1)}}{=} \lambda_{d-i} \frac{(\theta^*_{d-i} - \theta^*_{d-i-1}) \cdots (\theta^*_{d-i} - \theta^*_0)}{(\theta^*_{d-i+1} - \theta^*_{d-i}) \cdots (\theta^*_{d-i+1} - \theta^*_0)}$$

$$(A, A^*; \{V_{d-i}\}_{i=0}^d, \{V^*_{d-i}\}_{i=0}^d)$$

data $\left(\{\theta_{d-i}\}_{i=0}^d, \{\theta^*_{d-i}\}_{i=0}^d, \{\lambda_{d-i-1}\}_{i=0}^{d-1} \right)$

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(3) Use the L-system

$(A^*, A\ ;\ \{V_i^*\}_{i=0}^d,\ \{V_i\}_{i=0}^d)$

If the ordering of $V_0^*, V_1^*, \ldots, V_d^*$ is reversed,

$(A^*, A\ ;\ \{V_{d-i}^*\}_{i=0}^d,\ \{V_i\}_{i=0}^d)$ has data

$(\{\theta_{d-i}^*\}_{i=0}^d,\ \{\theta_i\}_{i=0}^d,\ \{\hat{\lambda}_{d-i-1}\}_{i=0}^d)$.

The claim for c_i^* follows from (1).

(4) Use the L-system

$(A^*, A\ ;\ \{V_i^*\}_{i=0}^d,\ \{V_i\}_{i=0}^d)$.

The claim for b_i^* follows from (2),

since $(A^*, A\ ;\ \{V_i^*\}_{i=0}^d,\ \{V_i\}_{i=0}^d)$ has

data $(\{\theta_i^*\}_{i=0}^d,\ \{\theta_i\}_{i=0}^d,\ \{\lambda_i\}_{i=0}^{d-1})$. ▨

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§6.3 Classification of dual systems of orth. polys.

$(A, A^*\ ;\ \{V_i\}_{i=0}^d,\ \{V_i^*\}_{i=0}^d)$

L-system on $V \simeq \mathbb{C}^{d+1}$

$\Downarrow$

$\underset{\text{diag.}}{D}$, $B^* \in M(\theta_0^*)$ via standard basis

L-pair on $\mathbb{C}^{d+1}$

$\simeq$

$B \in M(\theta_0)$, $\underset{\text{diag.}}{D^*}$ via dual standard basis

L-pair on $\mathbb{C}^{d+1}$

$\exists\ S$ nonsingular matrix

$S^{-1} B S = D$

$S B^* S^{-1} = D^*$

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$$\mathcal{M}(\theta_0) \ni B = \begin{pmatrix} a_0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{d-1} & a_{d-1} & b_{d-1} \\ 0 & & & c_d & a_d \end{pmatrix}$$

tridiagonal
irreducible
diagonalizable
row sum θ_0

$$\downarrow$$

$$\mathcal{P}(\theta_0) \ni \{p_i(x)\}_{i=0}^{d}$$

orthogonal polynomials
support $\{\theta_0, \theta_1, \cdots, \theta_d\}$ eigen values of B
$p_i(\theta_0) = 1, \quad 0 \le i \le d$

$$x\,p_i(x) = c_i\,p_{i-1}(x) + a_i\,p_i(x) + b_i\,p_{i+1}(x), \quad 0 \le i \le d$$

$$p_{-1}(x) = 0, \quad p_0(x) = 1, \quad b_d = 1$$

$$p_{d+1}(x) = \frac{1}{b_0 b_1 \cdots b_{d-1}} (x-\theta_0)(x-\theta_1)\cdots(x-\theta_d)$$

min. poly. of B

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$$\mathcal{M}(\theta_0^*) \ni B^* = \begin{pmatrix} a_0^* & b_0^* & & & 0 \\ c_1^* & a_1^* & b_1^* & & \\ & \ddots & \ddots & \ddots & \\ & & c_{d-1}^* & a_{d-1}^* & b_{d-1}^* \\ 0 & & & c_d^* & b_d^* \end{pmatrix}$$

$$\downarrow$$

$$\mathcal{P}(\theta_0^*) \ni \{p_i^*(x)\}_{i=0}^{d}$$

orthogonal polynomials
support $\{\theta_0^*, \theta_1^*, \cdots, \theta_d^*\}$
$p_i^*(\theta_0^*) = 1, \quad 0 \le i \le d$

$\{p_i(x)\}_{i=0}^{d}, \ \{p_i^*(x)\}_{i=0}^{d}$: <u>dual</u>

$$p_i(\theta_j) = p_j^*(\theta_i^*), \quad \forall i, j \in \{0, 1, \cdots, d\}$$

all dual systems of orthogonal polynomials arise in this way from L-pairs.

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$\left(A, A^* ; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d\right)$ L-system

data $\left(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\lambda_i\}_{i=0}^{d-1}\right)$

Theorem

(1)
$$p_i(x) = \sum_{\nu=0}^{i} \frac{(\theta_i^*-\theta_0^*)\cdots(\theta_i^*-\theta_{\nu-1}^*)}{\lambda_0 \cdots \lambda_{\nu-1}} (x-\theta_0)\cdots(x-\theta_{\nu-1}),$$
$$0 \le i \le d$$

(2)
$$p_i^*(x) = \sum_{\nu=0}^{i} \frac{(\theta_i-\theta_0)\cdots(\theta_i-\theta_{\nu-1})}{\lambda_0 \cdots \lambda_{\nu-1}} (x-\theta_0^*)\cdots(x-\theta_{\nu-1}^*),$$
$$0 \le i \le d$$

Proof

(1) Set
$$p_i(x) = \sum_{\nu=0}^{i} t_{\nu i} (x-\theta_0)\cdots(x-\theta_{\nu-1}), \quad 0 \le i \le d.$$

$V = \bigoplus_{i=0}^{d} U_i$ weight space decomposition

$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d)$

$F_i : V \longrightarrow U_i$ projection

$$\begin{cases} A = R + \sum_{i=0}^{d} \theta_i F_i, & R U_i = U_{i+1}, \quad 0 \le i \le d, \ U_{d+1} = 0 \\ A^* = L + \sum_{i=0}^{d} \theta_i^* F_i, & L U_i = U_{i-1}, \quad 0 \le i \le d, \ U_{-1} = 0 \end{cases}$$

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$U_0 \ni u_0 \neq 1.$

$U_i \ni u_i = R^i u_0 = (A-\theta_{i-1})\cdots(A-\theta_0) u_0$

$$p_i(A) u_0 = \sum_{\nu=0}^{i} t_{\nu i} u_\nu$$

Claim $p_i(A) u_0 \in V_i^*$

proof

$$k_0 = 1, \quad k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} = \frac{b_{i-1}}{c_i} k_{i-1}$$

Set $w_i = k_i p_i(A) u_0.$

Then

$$A p_i(A) u_0 = c_i p_{i-1}(A) u_0 + a_i p_i(A) u_0 + b_i p_{i+1}(A) u_0$$

$$A \frac{w_i}{k_i} = c_i \frac{w_{i-1}}{k_{i-1}} + a_i \frac{w_i}{k_i} + b_i \frac{w_{i+1}}{k_{i+1}}$$

$$A w_i = b_{i-1} w_{i-1} + a_i w_i + c_{i+1} w_{i+1}$$

$w_{-1} = 0, \quad w_0 = u_0 \in U_0$

This means

$w_0, w_1, \cdots, w_d$ are a dual standard basis.

In particular, $w_i \in V_i^*$. //

$$V_i^* \ni p_i(A)u_0 = \sum_{\nu=0}^{i} t_{\nu i} u_\nu$$

By Lemma on page 5–8

$$v_i^* = \sum_{\nu=0}^{i} \tilde{c}^*_{\nu i} u_\nu \quad \in V_i^*$$

$$\tilde{c}^*_{\nu i} = \frac{\lambda_\nu \cdots \lambda_{i-1}}{(\theta_i^* - \theta_\nu^*)\cdots(\theta_i^* - \theta_{i-1}^*)}$$

On the other hand,

$$t_{0i} = p_i(\theta_0) = 1$$

So

$$t_{\nu i} = \frac{\tilde{c}^*_{\nu i}}{\tilde{c}^*_{0i}} = \frac{(\theta_i^* - \theta_0^*)\cdots(\theta_i^* - \theta_{\nu-1}^*)}{\lambda_0 \cdots\cdots \lambda_{\nu-1}} \qquad \blacksquare$$

<u>type I</u> $\quad q \neq \pm 1$

$$\theta_i = \theta_0 + h\frac{1}{q^i}(1-q^i)(1-sq^{i+1}), \quad 0 \le i \le d$$

$$\theta_i^* = \theta_0^* + h^*\frac{1}{q^i}(1-q^i)(1-s^*q^{i+1}), \quad 0 \le i \le d$$

$$\lambda_i = hh^* q^{-2i-1}(1-q^{i+1})(1-q^{i-d})(1-r_1q^{i+1})(1-r_2q^{i+1}), \quad 0 \le i \le d-1$$

$$r_1r_2 = ss^*q^{d+1}$$

<u>Theorem</u>

(1) $$p_i(x) = {}_4\phi_3\left(\begin{matrix} q^{-i}, & s^*q^{i+1}, & q^{-y}, & sq^{y+1} \\ q^{-d}, & r_1q, & r_2q & \end{matrix}; q, q\right), \quad 0 \le i \le d$$

$$x = \xi(y) = \theta_0 + h\frac{1}{q^y}(1-q^y)(1-sq^{y+1})$$

$$b_i = \frac{h(1-q^{i-d})(1-s^*q^{i+1})(1-r_1q^{i+1})(1-r_2q^{i+1})}{(1-s^*q^{2i+2})(1-s^*q^{2i+1})}, \quad 0 \le i \le d-1$$

$$c_i = \frac{hq}{s^*}\frac{(1-q^i)(q^{-d-1}-s^*q^i)(r_1-s^*q^i)(r_2-s^*q^i)}{(1-s^*q^{2i})(1-s^*q^{2i+1})}, \quad 1 \le i \le d$$

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(2) $$p_i^*(x) = {}_4\phi_3\left(\begin{matrix} q^{-i}, & sq^{i+1}, & q^{-y}, & s^*q^{y+1} \\ q^{-d}, & r_1q, & r_2q & \end{matrix}; q, q\right), \quad 0 \le i \le d$$

$$x = \xi^*(y) = \theta_0^* + h^* \frac{1}{q^y}(1-q^y)(1-s^*q^{y+1})$$

$$b_i^* = \frac{h^*(1-q^{i-d})(1-sq^{i+1})(1-r_1q^{i+1})(1-r_2q^{i+1})}{(1-sq^{2i+2})(1-sq^{2i+1})}, \quad 0 \le i \le d-1$$

$$c_i^* = \frac{h^*q}{s}\,\frac{(1-q^i)(q^{-d-1}-sq^i)(r_1-sq^i)(r_2-sq^i)}{(1-sq^{2i})(1-sq^{2i+1})}, \quad 1 \le i \le d$$

type II $\quad q \to 1, \quad h(1-q)^2 \to h', \quad h^*(1-q)^2 \to h^{*\prime}$

$$\frac{1-s}{1-q} \to s', \quad \frac{1-s^*}{1-q} \to s^{*\prime}$$

$$\frac{1-r_1}{1-q} \to r_1', \quad \frac{1-r_2}{1-q} \to r_2'$$

type III $\quad q \to -1, \quad (1+q)h \to h', \quad (1+q)h^* \to h^{*\prime}$

$$\frac{1-s}{1+q} \to s', \quad \frac{1-s^*}{1+q} \to s^{*\prime}$$

$$\frac{1+r_1}{1+q} \to r_1', \quad \frac{1+r_2q^{d+1}}{1+q} \to r_2' + d + 1$$

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