Some Problems on Cayley Graphs

Elena Konstantinova

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Famnit Lecturs | Famnitova predavanja | 1

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Famnit Lectures 1 | ISSN 2335-3708

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Dr Elena Konstantinova
Sobolev Institute of Mathematics
Novosibirsk, Russia

Published by
University of Primorska Press,
Titov trg 4, 6000 Koper
Koper · 2013

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www.hippocampus.si
Print run · 50 · *Not for sale*

CIP – Kataložni zapis o publikaciji
Narodna in univerzitetna knjižnica, Ljubljana

519.17
KONSTANTINOVA, Elena
Some problems on Cayley graphs / Elena Konstantinova. – Koper :
University of Primorska Press, 2013. – (Famnit lectures =
Famnitova predavanja, ISSN 2335-3708 ; 1)

Način dostopa (URL): http://www.hippocampus.si/ISBN/978-961-6832-50-2.pdf
Način dostopa (URL): http://www.hippocampus.si/ISBN/978-961-6832-51-9/index.html

ISBN 978-961-6832-49-6
ISBN 978-961-6832-50-2 (pdf)
ISBN 978-961-6832-51-9 (html)
269231616

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Introduction

The definition of a Cayley graph was introduced by Arthur Cayley in 1878 to explain the concept of an abstract group generated by a set of generators. This definition has two main sources: group theory and graph theory.

Group theory studies algebraic structures known as groups. A group is a set of elements with an operation on these elements such that the set and the operation must satisfy to the group axioms, namely closure, associativity, identity and invertibility. The earliest study of groups as such probably goes back to the work of Lagrange in the late 18th century. However, this work was somewhat isolated. Publications of Augustin Louis Cauchy and Évariste Galois are more commonly referred to as the beginning of group theory. Évariste Galois, in the 1830s, was the first to employ groups to determine the solvability of polynomial equations. Arthur Cayley and Augustin Louis Cauchy pushed these investigations further by creating the theory of permutation group.

Graph theory studies discrete structures known as graphs to model pairwise relations between objects from a certain collection. So, a graph is a collection of vertices or nodes and a collection of edges that connect pairs of vertices. It is known that the first paper in the history of graph theory was written by Leonhard Euler on the Seven Bridges of Königsberg and published in 1736 [15]. The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other and the mainland by seven bridges. The problem was to find a walk through the city that would cross each bridge once and only once. Euler proved that the problem has no solution.

There is also a big branch of mathematics in which algebraic methods are applied to problems about graphs. This is algebraic graph theory

involving the use of group theory and the study of graph invariants. One of its branches also studies Cayley graphs with properties related to the structure of the group.

In the last fifty years, the theory of Cayley graphs has been developed to a rather big branch in algebraic graph theory. It has relations with many practical problems, and also with some classical problems in pure mathematics such as classification, isomorphism or enumeration problems. There are problems related to Cayley graphs which are interesting to graph and group theorists such as hamiltonian or diameter problems, and to computer scientists and molecular biologists such as sorting by reversals.

These Notes are based on the lectures I gave in May–June 2012 at University of Primorska FAMNIT in Koper, Slovenia, in the frame of "2012 Graph Theory Semester". They consist of selected problems on Cayley graphs. First of all, we pay attention to the hamiltonian and diameter problems for Cayley graphs. We discuss combinatorial conditions for hamiltonicity, Lovász and Babai conjectures, hamiltonicity of Cayley graphs on the symmetric group and hamiltonicity of Cayley graphs on finite groups with a small generating set. The diameter problem is considered for abelian and non–abelian groups (there is a fundamental difference for them), for the Pancake graphs (unburnt and burnt cases). The interesting fact is that the diameter problem appears in some popular games, for instance, in the Rubik's cube puzzle. Recently it was shown that every position of Rubik's cube can be solved in twenty moves or less, which represents the diameter of a corresponding Cayley graph. In general, computing the diameter of an arbitrary Cayley graph over a set of generators is NP–hard. We also emphasize the variety of applications of Cayley graphs in solving combinatorial and graph–theoretical problems and show how these graphs connect with applied problems in molecular biology and computer sciences. Special thanks go to Alexey Medvedev on helping in preparing pictures for these Lecture Notes.

Chapter 1

Definitions, basic properties and examples

In this section, basic definitions are given, notation is introduced and examples of graphs are presented. We also discuss some combinatorial and structural properties of Cayley graphs that we will be used later. For more definitions and details on graphs and groups we refer the reader to the books [13, 14, 18, 60].

1.1 Groups and graphs

Let G be a finite group. The elements of a subset S of a group G are called *generators* of G, and S is said to be a *generating set*, if every element of G can be expressed as a finite product of generators. We also say that G is generated by S. The identity of a group G will be denoted by e and the operation will be written as multiplication. A subset S of G is *identity free* if $e \notin S$ and it is *symmetric* (or closed under inverses) if $s \in S$ implies $s^{-1} \in S$. The last condition can be also denoted by $S = S^{-1}$, where $S^{-1} = \{s^{-1} : s \in S\}$.

A *permutation* π on the set $X = \{1, \ldots, n\}$ is a bijective mapping (i.e. one–to–one and surjective) from X to X. We write a permutation π in one–line notation as $\pi = [\pi_1 \pi_2 \ldots \pi_n]$ where $\pi_i = \pi(i)$ are images of elements for every $i \in \{1, \ldots, n\}$. The group of all permutations acting on the set $\{1, \ldots, n\}$ is called the *symmetric group* and denoted by Sym_n. The cardinality of the symmetric group Sym_n is defined by the number of all its elements, that is $|Sym_n| = n!$. In particular, the symmetric group Sym_n

is generated by transpositions swapping any two neighbors elements of a permutations that is the generating set $S = \{(1\,2), (2\,3), \ldots, (n-1\,n)\}$. This set of generators is also known as the set of the $(n-1)$ Coxeter generators of Sym_n and it is an important instance in combinatorics of Coxeter groups (for details see [16]).

Let $S \subset G$ be the identity free and symmetric generating set of a finite group G. In the *Cayley graph* $\Gamma = Cay(G, S) = (V, E)$ vertices correspond to the elements of the group, i.e. $V = G$, and edges correspond to multiplication on the right by generators, i.e. $E = \{\{g, gs\} : g \in G, s \in S\}$. The identity free condition is imposed so that there are no loops in Γ. The reason for the second condition is that an edge should be in the graph no matter which end vertex is used. So when there is an edge from g to gs, there is also an edge from gs to $(gs)s^{-1} = g$.

For example, if G is the symmetric group Sym_3, and the generating set S is presented by all transpositions from the set $T = \{(1\,2), (2\,3), (1\,3)\}$, then the Cayley graph $Sym_3(T) = Cay(Sym_3, T)$ is isomorphic to $K_{3,3}$ (see Figure 1). Here $K_{p,q}$ is the complete bipartite graph with p and q vertices in the two parts, respectively.

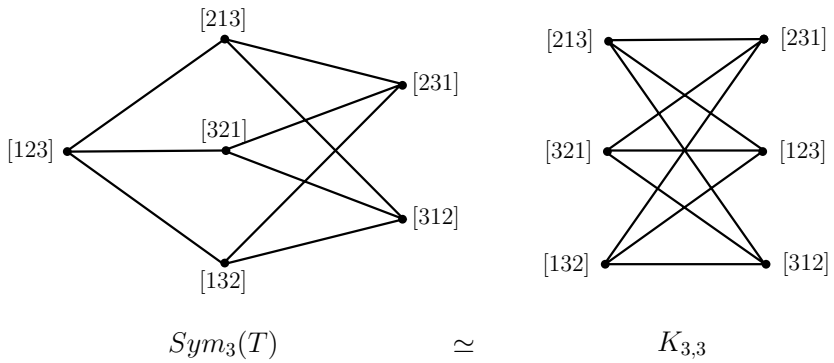


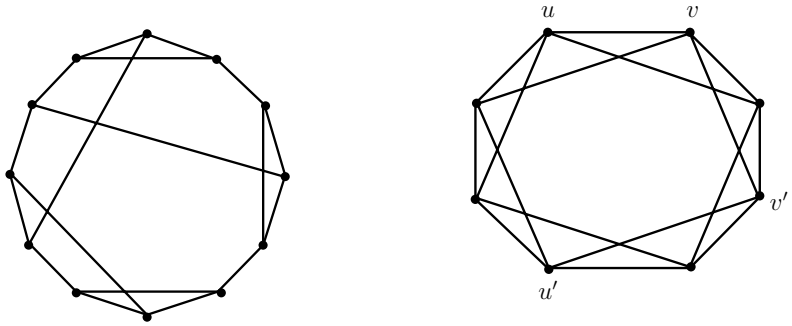
Figure 1. The Cayley graph $Sym_3(T)$ is isomorphic to $K_{3,3}$

Let us note here that if the symmetry condition doesn't hold in the definition of the Cayley graph then we have the Cayley digraphs which are not considered in these Lecture Notes.

1.2 Symmetry and regularity of graphs

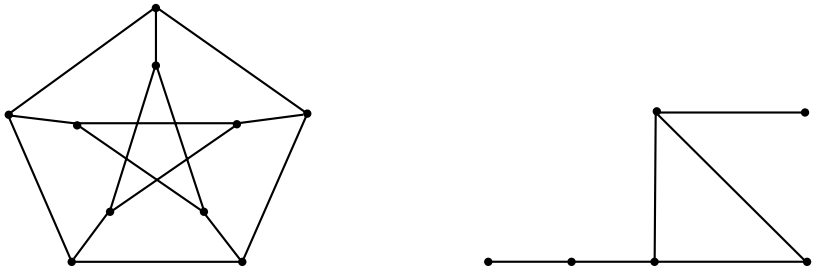
Let $\Gamma = (V, E)$ be a finite simple graph. A graph Γ is said to be *regular of degree* k, or k*–regular* if every vertex has degree k. A regular graph of degree 3 is called *cubic*.

A permutation σ of the vertex set of a graph Γ is called an *automorphism* provided that $\{u, v\}$ is an edge of Γ if and only if $\{\sigma(u), \sigma(v)\}$ is an edge of Γ. A graph Γ is said to be *vertex–transitive* if for any two vertices u and v of Γ, there is an automorphism σ of Γ satisfying $\sigma(u) = v$. Any vertex–transitive graph is a regular graph. However, not every regular graph is a vertex–transitive graph. For example, the Frucht graph is not vertex–transitive (see Figure 1.) A graph Γ is said to be *edge–transitive* if for any pair of edges x and y of Γ, there is an automorphism σ of Γ that maps x into y. These symmetry properties require that every vertex or every edge in a graph Γ looks the same and these two properties are not interchangeable. The graph Γ presented in Figure 2 is vertex–transitive but not edge–transitive since there is no an automorphism between edges $\{u, v\}$ and $\{u', v'\}$. The complete bipartite graph $K_{p,q}$, $p \neq q$, is the example of edge–transitive but not vertex–transitive graph.



The Frucht graph Γ

Figure 2. The Frucht graph: regular but not vertex–transitive; vertex–transitive but not edge–transitive graph Γ.



The Petersen graph Graph Γ

Figure 3.

Figure 3 presents the Petersen graph that is vertex–transitive and edge–transitive, and a graph Γ that is neither vertex– nor edge–transitive.

Proposition 1.2.1 *Let S be a set of generators for a group G. The Cayley graph $\Gamma = Cay(G, S)$ has the following properties:*
(i) it is a connected regular graph of degree equal to the cardinality of S;
(ii) it is a vertex–transitive graph.

Proof. Indeed, S is required to be a generating set of G so that Γ is connected. Since S is symmetric then every vertex in $\Gamma = Cay(G, S)$ has degree equal to $|S|$. Thus, the graph $\Gamma = Cay(G, S)$ is regular of degree equal to the cardinality of S. The Cayley graph $Cay(G, S)$ is vertex–transitive because the permutation $\sigma_g, g \in G$, defined by $\sigma_g(h) = gh$ for all $h \in G$ is an automorphism. □

Proposition 1.2.2 *Not every vertex–transitive graph is a Cayley graph.*

Proof. The simplest counterexample is the Petersen graph, which is a vertex–transitive but not a Cayley graph. The Petersen graph has order 10, it is cubic and its diameter is 2. This statement can be checked directly by examining of pairs (G, S) where G would have to be a group of order 10 and the size of S would have to be 3. There are only two nonisomorphic groups of order 10 and, checking all 3–sets S in each with the identity free and symmetric properties, one finds that each gives a diameter greater than 2. This proof was given by Biggs [14]. □

Denote by $d(u,v)$ the *path distance* between the vertices u and v in Γ, and by $diam(\Gamma) = \max\{d(u,v) : u,v \in V\}$ the *diameter* of Γ. In particular, in a Cayley graph the diameter is the maximum, over $g \in G$, of the length of a shortest expression for g as a product of generators. Let

$$S_r(v) = \{u \in V(\Gamma) : \ d(v,u) = r\} \text{ and } B_r(v) = \{u \in V(\Gamma) : \ d(v,u) \leqslant r\}$$

be the metric sphere and the metric ball of radius r centered at the vertex $v \in V(\Gamma)$, respectively. The vertices $u \in B_r(v)$ are *r–neighbors* of the vertex v. For $v \in V(\Gamma)$ we put $k_i(v) = |S_i(v)|$ and for $u \in S_i(v)$ we set

$$c_i(v,u) = |\{x \in S_{i-1}(v) \ : \ d(x,u) = 1\}|,$$

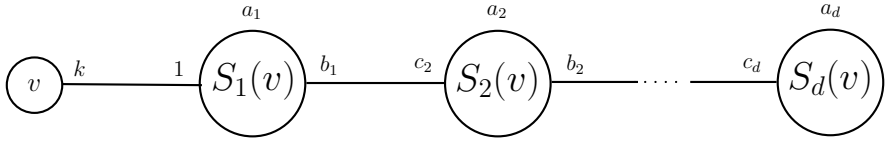
$$a_i(v,u) = |\{x \in S_i(v) \quad : \ d(x,u) = 1\}|,$$

$$b_i(v,u) = |\{x \in S_{i+1}(v) \ : \ d(x,u) = 1\}|.$$

From this $a_1(v,u) = a_1(u,v)$ is the number of triangles over the edge $\{v,u\}$ and $c_2(v,u)$ is the number of common neighbors of $v \in V$ and $u \in S_2(v)$. Let $\lambda = \lambda(\Gamma) = \max_{v\in V,\ u\in S_1(v)} a_1(v,u)$ and $\mu = \mu(\Gamma) = \max_{v\in V,\ u\in S_2(v)} c_2(v,u)$.

A simple connected graph Γ is *distance–regular* if there are integers b_i, c_i for $i \geqslant 0$ such that for any two vertices v and u at distance $d(v,u) = i$ there are precisely c_i neighbors of u in $S_{i-1}(v)$ and b_i neighbors of u in $S_{i+1}(v)$. Evidently Γ is regular of valency $k = b_0$, or *k–regular*. The numbers c_i, b_i and $a_i = k - b_i - c_i$, $i = 0, \ldots, d$, where $d = diam(\Gamma)$ is the diameter of Γ, are called the *intersection numbers* of Γ and the sequence $(b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d)$ is called the *intersection array* of Γ.

The schematic representation of the intersection array for a distance–regular graph is given below:



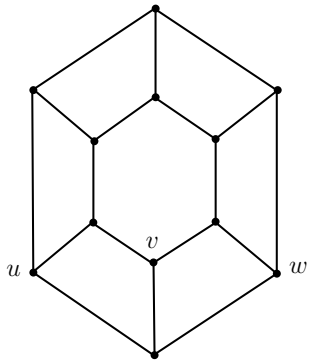


Figure 4. The cyclic 6–ladder L_6

A k–regular simple graph Γ is *strongly regular* if there exist integers λ and μ such that every adjacent pair of vertices has λ common neighbors, and every nonadjacent pair of vertices has μ common neighbors. A simple connected graph Γ is *distance–transitive* if, for any two arbitrary–chosen pairs of vertices (v, u) and (v', u') at the same distance $d(v, u) = d(v', u')$, there is an automorphism σ of Γ satisfying $\sigma(v) = v'$ and $\sigma(u) = u'$.

Proposition 1.2.3 *Any distance–transitive graph is vertex–transitive.*

This statement is obvious (consider two vertices at distance 0). However, the converse is not true in general. There exist vertex–transitive graphs that are not distance–transitive. For instance, the cyclic 6–ladder L_6 is clearly vertex–transitive – we can rotate and reflect it (see Figure 4). However, it is not distance–transitive since there are two pairs of vertices u, v and u, w at distance two, i.e. $d(u, v) = d(u, w) = 2$, such that there is no automorphism that moves one pair of vertices into the other, as there is only one path between vertices u, w, while there are two paths for the vertices u, v.

The following graphs are distance–transitive: the complete graphs K_n; the cycles C_n; the platonic graphs that are obtained from the five Platonic solids: their vertices and edges form distance–regular and distance–transitive graphs as well with intersection arrays $\{3; 1\}$ for tetrahedron, $\{4, 1; 1, 4\}$ for octahedron, $\{3, 2, 1; 1, 2, 3\}$ for cube, $\{5, 2, 1; 1, 2, 5\}$ for icosahedron, $\{3, 2, 1, 1, 1; 1, 1, 1, 2, 3\}$ for dodecahedron; and many others.

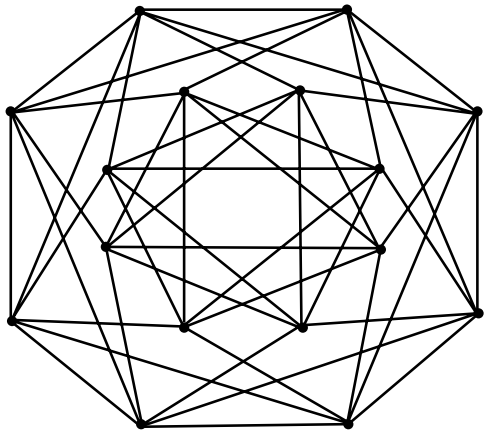


Figure 5. The Shrikhande graph: distance–regular, but not distance–transitive

Proposition 1.2.4 *Any distance–transitive graph is distance–regular.*

This statement is also obvious. The converse is not necessarily true. The smallest distance–regular graph that is not distance–transitive is the Shrikhande graph presented in Figure 5 (for details we refer to [17]).

An example of a distance–regular but not distance–transitive or vertex–transitive graph is the Adel'son–Vel'skii graph whose vertices are the 26 symbols x_i, y_i, $i \in \mathbb{Z}_{13}$ ($\mathbb{Z}_{13}$ is the additive group of integers modulo 13) and in which the following vertices are adjacent:

1. x_i adjacent $x_j \iff |i-j| = 1, 3, 4$.
2. y_i adjacent $y_j \iff |i-j| = 2, 5, 6$.
3. x_i adjacent $y_j \iff i-j = 0, 1, 3, 9$.

(all taken modulo 13). This graph is distance–regular with the intersection array $\{10, 6; 1, 4\}$. However, it is not distance–transitive or even vertex–transitive: there is no automorphism taking any x_i to any y_i.

1.3 Examples

1.3.1 Some families of Cayley graphs

The *complete* graph K_n is a Cayley graph on the additive group $\mathbb{Z}_n$ of integers modulo n with generating set of all non–zero elements of $\mathbb{Z}_n$.

The *circulant* is the Cayley graph $Cay(\mathbb{Z}_n, S)$ where $S \subset \mathbb{Z}_n$ is an arbitrary generating set. The most prominent example is the *cycle* C_n.

The *multidimensional torus* $T_{n,k}$, $n \geqslant 2\,, k \geqslant 2\,$, is the cartesian product of n cycles of length k. It has k^n vertices of degree $2n$ and its diameter is $n\lceil\frac{k}{2}\rceil$. It is the Cayley graph of the group $\mathbb{Z}_k^n$ that is the direct product of $\mathbb{Z}_k$ with itself n times, which is generated by $2n$ generators from the set $S = \{(\underbrace{0,\ldots,0}_{i},1,\underbrace{0,\ldots,0}_{n-i-1}), (\underbrace{0,\ldots,0}_{i},-1,\underbrace{0,\ldots,0}_{n-i-1}),\ 0 \leqslant i \leqslant n-1\}$.

The *hypercube* (or *n–dimensional cube*) H_n is the graph with vertex set $\{x_1x_2\ldots x_n : x_i \in \{0,1\}\}$ in which two vertices $(v_1v_2\ldots v_n)$ and $(u_1u_2\ldots u_n)$ are adjacent if and only if $v_i = u_i$ for all but one $i\,, 1 \leqslant i \leqslant n$. It is a distance–transitive graph with diameter and degree of n and can be considered as a particular case of torus, namely $T_{n,2}$, since it is the cartesian product of n complete graphs K_2. It is the Cayley graph on the group $\mathbb{Z}_2^n$ with the generating set $S = \{(\underbrace{0,\ldots,0}_{i},1,\underbrace{0,\ldots,0}_{n-i-1}), 0 \leqslant i \leqslant n-1\}$.

The *butterfly graph* BF_n is the Cayley graph with vertex set $V = \mathbb{Z}_n \times \mathbb{Z}_2^n$, $|V| = n \cdot 2^n$, and with edges defined as follows. Any vertex $(i,x) \in V$, where $0 \leqslant i \leqslant n-1$ and $x = (x_0x_1\ldots x_{n-1})$, is connected to $(i+1,x)$ and $(i+1,x(i))$ where $x(i)$ denotes the string, which is derived from x by replacing x_i by $1-x_i$. All arithmetic on indices i is assumed to be modulo n. Thus, BF_n is derived from H_n by replacing each vertex x by a cycle of length n, however the vertices of this cycle are connected to vertices of other cycles in a different way such that the degree is 4 (for $n \geqslant 3$). For example, $BF_2 = H_3$ and $BF_1 = K_2$. The diameter of BF_n is $\lfloor\frac{3n}{2}\rfloor$. This graph is not edge–transitive, not distance–regular and hence not distance–transitive. This graph is also the Cayley graph on the subgroup of Sym_{2n} of $n2^n$ elements generated by $(12\ldots2n)^2$ and $(12\ldots2n)^2(12)$.

1.3.2 Hamming graph: distance–transitive Cayley graph

Let F_q^n be the Hamming space (where F_q is the field of q elements) consisting of the q^n vectors (or words) of length n over the alphabet $\{0, 1, ..., q-1\}$, $q \geqslant 2$. This space is endowed with the Hamming distance $d(x, y)$ which equals to the number of coordinate positions in which x and y differ. This space can be viewed as the *Hamming graph* $L_n(q)$ with vertex set given by the vector space F_q^n where $\{x, y\}$ is an edge of $L_n(q)$ if and only if $d(x, y) = 1$. The Hamming graph $L_n(q)$ is, equivalently, the cartesian product of n complete graphs K_q. This graph has the following properties:

1. its diameter is n;
2. it is distance–transitive and Cayley graph;
3. it has intersection array given by $b_j = (n-j)(q-1)$ and $c_j = j$ for $0 \leqslant j \leqslant n$.

Indeed, the Hamming graph is the Cayley graph on the additive group F_q^n when we take the generating set $S = \{xe_i : x \in (F_q)^\times, 1 \leqslant i \leqslant n\}$ where $(F_q)^\times$ is the cartesian product of F_q and $e_i = (0, ..., 0, 1, 0, ...0)$ are the standard basis vectors of F_q^n.

For the particular case $n = 2$ the Hamming graph $L_2(q)$ is also known as the *lattice graph* over F_q. This graph is strongly regular with parameters $|V(L_2(q))| = q^2$, $k = 2(q-1)$, $\lambda = q-2$, $\mu = 2$. The lattice graph $L_2(3)$ is presented in Figure 6.

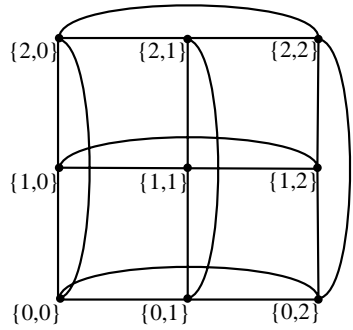


Figure 6. The lattice graph $L_2(3)$

1.3.3 Johnson graph: distance–transitive not Cayley graph

The Johnson graph $J(n, m)$ is defined on the vertex set of the m–element subsets of an n–element set. Two vertices are adjacent when they meet in a $(m-1)$–element set. On $J(n, m)$ the Johnson distance is defined as half the (even) Hamming distance, and two vertices x, y are joined by an edge if and only if they are at Johnson distance one from each other. This graph has the following properties:

1. its diameter d is $min\{m, n-m\}$;
2. it is distance–transitive but not Cayley graph (for $m > 2$);
3. it has intersection array given by $b_j = (m-j)(n-m-j)$ and $c_j = j^2$ for $0 \leqslant j \leqslant d$.

Let us note that $J(n, 1) \cong K_n$ and it is a Cayley graph. In general, to show that the Johnson graph is not a Cayley graph just take n or $n-1$ congruent to 2(mod 4). Then $\binom{n}{2}$ is odd, etc.

In the particular case $m = 2$ and $n \geqslant 4$ the Johnson graph $J(n, 2)$ is known as the *triangular graph* $T(n)$. As vertices it has the 2–element subsets of an n–set and two vertices are adjacent if and only if they are not disjoint. This graph is strongly regular with parameters $|V(T(n))| = \frac{n(n-1)}{2}$, $k = 2(n-2)$, $\lambda = n-2$, $\mu = 4$. The triangular graph $T(4)$ is presented in Figure 7.

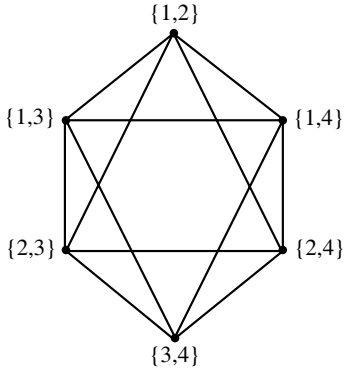


Figure 7. The triangular graph $T(4)$

1.3.4 Kneser graph: when it is a Cayley graph

The Kneser graph $K(n,k)$, $k \geqslant 1$, $n \geqslant 2k+1$, is the graph whose vertices correspond to the k–element subsets of a set of n elements, and where two vertices are connected if and only if two corresponding sets are disjoint. The graph $K(2n-1, n-1)$ is also referred as the *odd graph* O_n. The complete graph K_n on n vertices is $K(n,1)$, and the Petersen graph is $K(5,2)$ (see Figure 8). The graph $K(n,k)$ has the following properties:

1. its diameter is $\lceil \frac{k-1}{n-2k} \rceil + 1$;
2. it is vertex–transitive and edge–transitive graph;
3. it is $\binom{n-k}{k}$–regular graph;
4. it is not, in general, a strongly regular graph.

The following theorem of Godsil provides the conditions on k which imply that the corresponding Kneser graph is a Cayley graph.

Theorem 1.3.1 [35] *Except the following cases, the Kneser graph* $K(n,k)$ *is not a Cayley graph:*
(i) $k = 2$, n *is a prime power and* $n \equiv 3 \bmod 4$;
(ii) $k = 3$, $n = 8$ *or* 32.

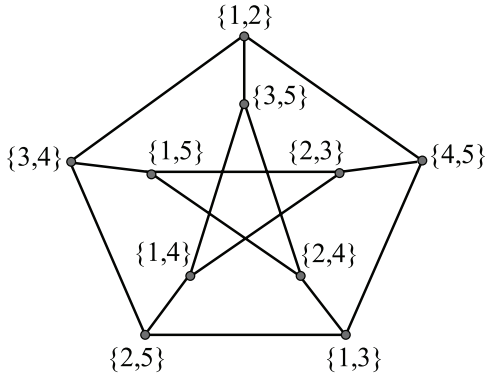


Figure 8. The graph $K(5,2)$ is isomorphic to the Petersen graph

1.3.5 Cayley graphs on the symmetric group

In this section we present Cayley graphs on the symmetric group Sym_n that are applied in computer science, molecular biology and coding theory.

The *Transposition graph* $Sym_n(T)$ on Sym_n is generated by transpositions from the set $T = \{t_{i,j} \in Sym_n, 1 \leqslant i < j \leqslant n\}$, where $t_{i,j}$ transposes the i–th and j–th elements of a permutation when multiplied on the right. The distance in this graph is defined as the least number of transpositions transforming one permutation into another. The transposition graph $Sym_n(T), n \geqslant 3$, has the following properties:

1. it is a connected graph of order $n!$ and diameter $n-1$;
2. it is bipartite $\binom{n}{2}$–regular graph;
3. it has no subgraphs isomorphic to $K_{2,4}$;
4. it is edge–transitive but not distance–regular and hence not distance–transitive (for $n > 3$).

This graph arises in molecular biology for analyzing transposons (genetic transpositions) that are mutations transferring a chromosomal segment to a new position on the same or another chromosome [69, 74]. It is also considered in coding theory for solving a reconstruction problem [61].

The *Bubble–sort graph* $Sym_n(t)$, $n \geqslant 3$, on Sym_n is generated by transpositions from the set $t = \{t_{i,i+1} \in Sym_n, 1 \leqslant i < n\}$, where $t_{i,i+1}$ are 2–cycles interchanging i–th and $(i+1)$–th elements of a permutation when multiplied on the right. This graph has the following properties:

1. it is a connected graph of order $n!$ and diameter $\binom{n}{2}$;
2. it is bipartite $(n-1)$–regular graph;
3. it has no subgraphs isomorphic to $K_{2,3}$;
4. it is edge–transitive but not distance–regular and hence not distance–transitive (for $n > 3$).

This graph arises in computer programming [58] and in computer science to represent interconnection networks [39].

The *Star graph* $S_n = Sym_n(st)$ on Sym_n is generated by transpositions from the set $st = \{t_{1,i} \in Sym_n, 1 < i \leqslant n\}$ with the following properties:

1. it is a connected graph of order $n!$ and diameter $\lfloor \frac{3(n-1)}{2} \rfloor$;
2. it is bipartite $(n-1)$–regular graph;
3. it has no odd cycles but has even cycles of lengths $2\,l$, where $3 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor$;
4. it is edge–transitive but not distance–regular and hence not distance–transitive (for $n > 3$).

This graph is one of the most investigated in the theory of interconnection networks since many parallel algorithms are efficiently mapped on this graph (see [1, 39, 59]). In [1] it is shown that the diameter of the Star graph is $\lfloor \frac{3(n-1)}{2} \rfloor$. Moreover,

$$diam(S_n) = \begin{cases} \frac{3(n-1)}{2}, & \text{if } n \text{ odd}, \\ 1 + \frac{3(n-2)}{2}, & \text{if } n > 3 \text{ even}. \end{cases}$$

This is shown by considering the worst cases in a special sorting algorithm and then by giving permutations requiring at least $\lfloor \frac{3(n-1)}{2} \rfloor$ steps for this algorithm. For odd n, the permutation $\pi = [13254 \ldots n\, n-1]$ is considered. It is clear that three steps are needed to sort each pair $(i\, i-1)$ in π. Since there are $\frac{(n-1)}{2}$ such pairs, hence the above diameter follows. Likewise, for even n, the permutation $\pi = [214365 \ldots n\, n-1]$ is considered. One can swap position 2 with one, but the remaining $\frac{(n-2)}{2}$ pairs again each require three. Thus, the indicated diameter again follows.

The *Reversal graph* $Sym_n(R)$ is defined on Sym_n and generated by the reversals from the set $R = \{r_{i,j} \in Sym_n, 1 \leqslant i < j \leqslant n\}$ where a reversal $r_{i,j}$ is the operation of reversing segments $[i, j]$, $1 \leqslant i < j \leqslant n$, of a permutation when multiplied on the right, i.e. $[\ldots \pi_i\, \pi_{i+1} \ldots \pi_{j-1}\, \pi_j \ldots] r_{i,j} = [\ldots \pi_j\, \pi_{j-1} \ldots \pi_{i+1}\, \pi_i \ldots]$. For $n \geqslant 3$ this graph has the following properties:

1. it is a connected graph of order $n!$ and diameter $n-1$;
2. it is $\binom{n}{2}$–regular graph;
3. it has no contain triangles nor subgraphs isomorphic to $K_{2,4}$;

4. it is not edge–transitive, not distance–regular and hence not distance–transitive (for $n > 3$).

The reversal distance between two permutations in this graph, that is the least number of reversals needed to transform one permutation into another, corresponds to the reversal mutations in molecular biology. Reversal distance measures the amount of evolution that must have taken place at the chromosome level, assuming evolution proceeded by inversion. The analysis of genomes evolving by inversions leads to the combinatorial problem of sorting by reversals (for more details see Section 3.4). This graph also appears in coding theory in solving a reconstruction problem [49, 50, 52].

The Pancake graph $P_n = Cay(Sym_n, PR)$, $n \geqslant 2$, is the Cayley graph on Sym_n with the generating set $PR = \{r_i \in Sym_n, 2 \leqslant i \leqslant n\}$ of all prefix–reversals r_i reversing the order of any substring $[1, i]$, $2 \leqslant i \leqslant n$, of a permutation π when multiplied on the right, i.e. $[\pi_1 \ldots \pi_i \, \pi_{i+1} \ldots \pi_n] \, r_i = [\pi_i \ldots \pi_1 \, \pi_{i+1} \ldots \pi_n]$. For $n \geqslant 3$ this graph has the following properties:

1. it is a connected graph of order $n!$;

2. it is $(n-1)$–regular graph;

3. it has cycles of length $6 \leqslant l \leqslant n!$, but has no cycles of lengths 3,4,5;

4. it is not edge–transitive, not distance–regular and hence not distance–transitive (for $n \geqslant 4$).

This graph is well known because of the combinatorial *Pancake problem* which was posed in [28] as the problem of finding its diameter. The problem is still open. Some upper and lower bounds [33, 40] as well as exact values for $2 \leqslant n \leqslant 19$ are known [4, 19]. One of the main difficulties in solving this problem is a complicated cycle structure of the Pancake graph. As it was shown in [43, 76], all cycles of length l, where $6 \leqslant l \leqslant n!$, can be embedded in the Pancake graph P_n, $n \geqslant 3$. In particular, the graph is a Hamiltonian [81]. We will discuss all these problems on the Pancake graph in the next sections.

Chapter 2

Hamiltonicity of Cayley graphs

Let $\Gamma = (V, E)$ be a connected graph where $V = \{v_1, v_2, \ldots, v_n\}$. A *Hamiltonian cycle* in Γ is a spanning cycle $(v_1 v_2 \ldots v_n v_1)$ that visits each vertex exactly once, and a *Hamiltonian path* is a path $(v_1 v_2 \ldots v_n)$. A graph is *Hamiltonian* if it contains a Hamiltonian cycle. The Hamiltonicity problem, that is to check whether a graph is a Hamiltonian, was stated by Sir W.R. Hamilton in the 1850s (see [37]). Studying the Hamiltonian property of graphs is a favorite problem for graph and group theorists. Testing whether a graph is Hamiltonian is an NP–complete problem [32]. Hamiltonian paths and cycles naturally arise in computer science (see [59]), in the study of word–hyperbolic groups and automatic groups (see [29]), and in combinatorial designs (see [27]). For example, Hamiltonicity of the hypercube H_n is connected to a Gray code.

2.1 Hypercube graphs and a Gray code

A hypercube graph $H_n = L_n(2)$ is a particular case of the Hamming graph (see Section 1.3.2). It is a n–regular graph with 2^n vertices presented by vectors of length n. Two vertices are adjacent if and only if the corresponding vectors differ exactly in one position. The hypercube graph H_n is, equivalently, the cartesian product of n two–vertex complete graphs K_2. It is also a Cayley graph on the finite additive group $\mathbb{Z}_2^n$ with the generating set $S = \{(\underbrace{0, \ldots, 0}_{i}, 1, \underbrace{0, \ldots, 0}_{n-i-1}), 0 \leqslant i \leqslant n-1\}$, where $|S| = n$.

It is well known fact that every hypercube graph H_n is Hamiltonian for $n > 1$, and any Hamiltonian cycle of a labeled hypercube graph defines a

Gray code [77]. More precisely there is a bijective correspondence between the set of n–bit cyclic Gray codes and the set of Hamiltonian cycles in the hypercube H_n. By the definition, the *reflected binary code*, also known as a *Gray code* after Frank Gray, is a binary numeral system where two successive values differ in only one bit.

The Gray code list for n bits can be generated recursively from the list for $n-1$ bits by reflecting the list (i.e. listing the entries in reverse order), concatenating the original list with the reversed list, prefixing the entries in the original list with a binary 0, and then prefixing the entries in the reflected list with a binary 1. For example, generating the $n = 3$ list from the $n = 2$ list we have:

STEP 1.
2–bit list: $00, 01, 11, 10$;
reflected : $10, 11, 01, 00$;

STEP 2.
prefix old entries with 0 : $000, 001, 011, 010$;
prefix new entries with 1: $110, 111, 101, 100$;

STEP 3.
concatenated: $000, 001, 011, 010, 110, 111, 101, 100$.

So, for $n = 3$ the Gray code can be also presented as:

$$000\ 001\ 011\ 010\,|\,110\ 111\ 101\ 100.$$

and for $n = 4$ the Gray code is presented as:

0000 0001 0011 0010 0110 0111 0101 0100 | 1100 1101 1111 1110 1010 1011 1001 1000.

By the definition, Gray codes define the set of vectors of the hypercube graphs such that two successive vectors differ in only one bit. Hence, Gray codes correspond to the Hamiltonian path in the hypercube graphs. Moreover, since the first and last vectors also differ in one position we actually have the Hamiltonian cycles. The hypercube graphs H_2, H_3, H_4 and their Hamiltonian cycles are presented in Figure 9.

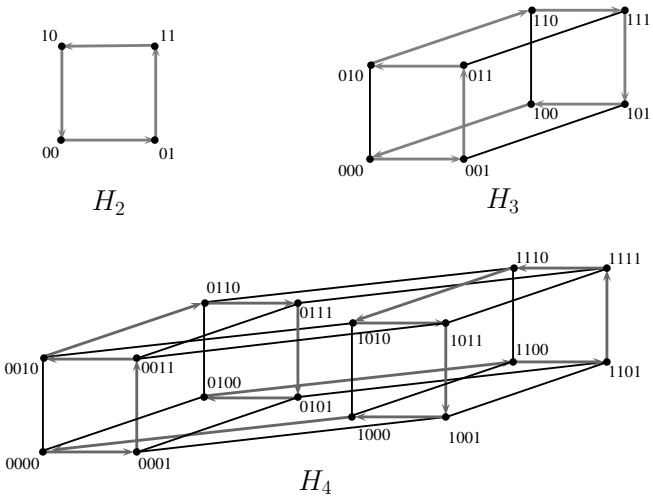


Figure 9. The hypercubes H_2, H_3, H_4 and their Hamiltonian cycles.

Now let us consider combinatorial conditions for Hamiltonicity of Cayley graphs obtained in the middle of 20th century and presented in the modern style of proofs by Igor Pak and Radoš Radoičić [68].

2.2 Combinatorial conditions for Hamiltonicity

It seems that the problem of finding Hamiltonian cycles in Cayley graphs was suggested for the first time by Rapaport–Strasser in 1959 [72]. Let G be a finite group with a generating set S and $|S| \leqslant 3$. In this section we consider simple relations on generators which suffice to prove that the Cayley graph $\Gamma = (G, S)$ contains a Hamiltonian cycle.

An element $\alpha \in G$ is called an *involution*, if $\alpha^2 = 1$.

Theorem 2.2.1 [72] *Let G be a finite group generated by three involutions α, β, γ such that $\alpha\beta = \beta\alpha$. Then the Cayley graph $\Gamma = Cay(G, \{\alpha, \beta, \gamma\})$ has a Hamiltonian cycle.*

Proof. For every $z \in G$ and every $X \subset G$, denote

$$\vartheta_z(X) = \{g \in G \setminus X : \ g = xz, \ x \in X\}.$$

Denote by $H = \langle \beta, \gamma \rangle$ a subgroup of G of order $|H| = 2m$. Let $X_1 = H$. Since H is a dihedral group, X_1 contains a Hamiltonian cycle:

$$1 \to \beta \to \beta\gamma \to \beta\gamma\beta \to \ldots \to (\beta\gamma)^{m-1}\beta \to (\beta\gamma)^m = 1 \qquad (2.1)$$

We shall construct a Hamiltonian cycle in Γ by induction. At step i we obtain a cycle which spans set $X_i \subset G$. Further, each X_i will satisfy the condition $\vartheta_\beta(X_i) = \vartheta_\gamma(X_i) = \emptyset$. This is equivalent to saying that each X_i is a union of left cosets of H in G, where *a left coset of H in G* is the set $gH = \{gh \mid h \in H\}$ with $g \in G$. By definition, $\vartheta_\beta(X_1) = \vartheta_\gamma(X_1) = \emptyset$. This establishes the base of induction.

Now suppose X_i is as above. Then either $\vartheta_\alpha(X_i) = \emptyset$, in which case the spanning cycle in $X_i = G$ is the desired Hamiltonian cycle. Otherwise, there exist $y \in \vartheta_\alpha(X_i) \subset G \setminus X_i$. Observe that $yH \cap X_i = \emptyset$, since otherwise $yh = x \in X_i$ for some $h \in H$. This implies that $y = xh^{-1} \in X_i$, since $h \in \langle \beta, \gamma \rangle$ and $z\beta, z\gamma \in X$ for all $z \in X$.

Let $X_{i+1} = X_i \cup yH$. Clearly, $\vartheta_\beta(X_{i+1}) = \vartheta_\gamma(X_{i+1}) = \emptyset$. By inductive assumption, $x = y\alpha \in X_i$ lies on a cycle which spans X_i. Then x must be connected to $x\beta$ and $x\gamma$, as $x\alpha^{-1} = y \notin X_i$. Consider a cycle in yH, obtained by multiplying cycle in (2.1) by y. Recall that $\alpha\beta = \beta\alpha$. This implies $x\beta\alpha = y\beta$. Remove edges $\{x, x\beta\}$ and $\{y, y\beta\}$ from cycles in X_i and yH, and add edges $\{x, y\}$ and $\{x\beta, y\beta\}$. This gives a cycle which spans X_{i+1}, and complete the proof. □

As an example, let us consider $G = Sym_{2n+1}$ and three involutions

$$\alpha = (12),$$

$$\beta = (12)(34)\cdots(2n-1\,2n),$$

$$\gamma = (23)(45)\cdots(2n\,2n+1)$$

(we use cycle notation here). Observe that

$$\beta\,\gamma = (135\ldots 2n-1\,2n+1\,2n\,2n-2\ldots 42),$$

so $\langle \alpha, \beta, \gamma \rangle = Sym_{2n+1}$. Note that $\alpha\beta = \beta\alpha$. Then Theorem 2.2.1 implies that the Cayley graph $\Gamma = Cay(G, \{\alpha, \beta, \gamma\})$ has a Hamiltonian cycle.

Cayley graphs on finite groups generated by two elements were considered by Rankin [71] in 1966. He obtained the following result.

Theorem 2.2.2 [71] *Let G be a finite group generated by two elements α, β such that $(\alpha\beta)^2 = 1$. Then the Cayley graph $\Gamma = Cay(G, \{\alpha, \beta\})$ has a Hamiltonian cycle.*

Proof. Again, we use the same inductive assumption as in Theorem 2.2.1. Moreover, we need a new simple label condition. Let $H = \langle\beta\rangle$, $X_1 = H$, and assume that $\vartheta_\alpha(X_i) = \vartheta_{\alpha^{-1}}(X_i) = \emptyset$. We also assume, by induction, that restriction of Γ to X_i contains an oriented Hamiltonian cycle C_i which contains only labels β and α^{-1}. We call these the *label conditions*.

The base of induction is obvious, namely $\vartheta_\alpha(X_1) = \vartheta_{\alpha^{-1}}(X_1) = \emptyset$.

For the step of induction, consider $y = x\alpha \in \vartheta_\alpha(X_i) \setminus X_i$. Note that the edge oriented towards $x \in X_i$ in C_i cannot have label α^{-1} (otherwise it is $\{y, x\}$ whereas $y \notin X_i$), nor labels α or β^{-1} (by the label conditions). Therefore, this edge has the only remaining label β, and $\{x\beta^{-1}, x\} \in C_i$.

Now consider a cycle R on yH with labels β on all edges, and observe that

$$x \to x\alpha = y \to x\alpha\beta = y\beta \to x\beta^{-1} = x\alpha\beta\alpha \to x$$

is a square which connects R and C_i. Formally, let

$$C_{i+1} = C_i \cup R + \{x, y\} + \{y\beta, x\beta^{-1}\} - \{x\beta^{-1}, x\} - \{y, y\beta\}$$

and observe that C_i is a Hamiltonian cycle on $X_{i+1} = X_i \cup yH$. Let C_{i+1} inherit the orientation from C_i and check that now C_{i+1} satisfies the label conditions with respect to the orientation.

In the case when $y = x\alpha^{-1} \notin X_i$, we consider the edge leaving $x \in X_i$ and proceed verbatim. If $\vartheta_\alpha(X_i) = \vartheta_{\alpha^{-1}}(X_i) = \emptyset$, we have $X_i = G$ which completes the proof. □

As an example, let us consider $G = Sym_n$, $\alpha = (12\ldots n)$, $\beta = (23\ldots n)$. Then $\alpha\beta^{-1} = (1\,n)$ is an involution, and by Theorem 2.2.2 the Cayley graph $\Gamma = Cay(G, \{\alpha, \beta\})$ has a Hamiltonian cycle.

The both theorems are presented here with respect to the proof given by Pak and Radoičić [68]. In Section 2.4 we will use these proofs to show the result by Pak and Radoičić on Hamiltonicity of Cayley graphs on finite groups with a small generating set.

2.3 Lovász and Babai conjectures

There is a famous Hamiltonicity problem for vertex–transitive graphs which was posed by László Lovász in 1970 and well–known as follows.

Problem 2.3.1 *Does every connected vertex–transitive graph with more than two vertices have a Hamiltonian path?*

To be more precisely he stated a research problem in [63] asking how one can

> *" ... construct a finite connected undirected graph which is symmetric and has no simple path containing all the vertices. A graph is symmetric if for any two vertices x and y it has an automorphism mapping x onto y."*

However, traditionally (see [25]) the problem is formulated in the positive and considered as the Lovász conjecture that every vertex–transitive graph has a Hamiltonian path.

There are only four vertex–transitive graphs on more than two vertices which do not have a Hamiltonian cycle, and all of these graphs have a Hamiltonian path. They are the Petersen graph, the Coxeter graph (it is a unique cubic distance–regular graph with intersection array $\{3, 2, 2, 1; 1, 1, 1, 2\}$ on 28 vertices and 42 edges presented in Figure 10) and graphs obtained from each of these two graphs by replacing each vertex with a triangle and joining the vertices in a natural way. In particular, it is unknown of a vertex–transitive graph without a Hamiltonian path. Furthermore, it was noted that all of the above four graphs are not Cayley graphs. So several people made the following conjecture.

Conjecture 2.3.2 *Every connected Cayley graph on a finite group has a Hamiltonian cycle.*

However, there is no consensus among experts what the answer on the problem above will be. In particular, Bojan Mohar and Laszlo Babai both made conjectures which are sharply critical of the Lovász problem. In 1996 Babai [6] made the following conjecture.

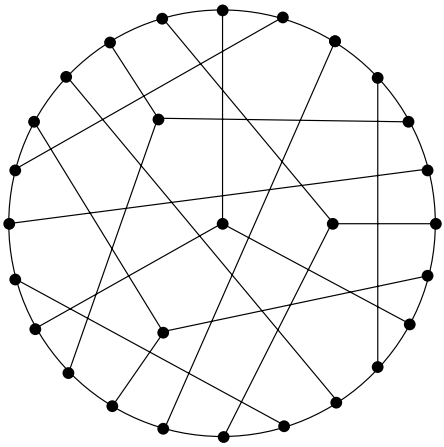


Figure 10. The Coxeter graph

Conjecture 2.3.3 [6] *For some $\varepsilon > o$, there exist infinitely many connected vertex–transitive graphs (even Cayley graphs) Γ without cycles of length $\geqslant (1-\varepsilon)|V(\Gamma)|$.*

Later Mohar [66] investigated the matching polynomial $\mu(\Gamma, x)$ of a graph Γ on n vertices defined as $\mu(\Gamma, x) = \sum_0^{[n/2]} (-1)^k p(\Gamma, k) x^{n-2k}$, where $p(\Gamma, k)$ is the number of k–matching in Γ, and formulated the following conjecture.

Conjecture 2.3.4 [66] *For every integer p there exists a vertex–transitive graph whose matching polynomial has a root of multiplicity at least p.*

It is known (see [36]) that a graph whose matching polynomial has a nonsimple root has no a Hamiltonian path. Hence, if such a vertex–transitive graph exists then Lovász conjecture will be disproved.

All these conjectures are still open. Most results obtained so far about the first conjecture on Cayley graphs were surveyed in 1996 by S. J. Curran and J. A. Gallian in [25] for abelian and dihedral groups, for groups of special orders, and certain extensions.

Let us recall that an *abelian group* is a group such that the order in which the binary operation is performed doesn't matter, and the *dihedral group* of order $2n$ is the abstract group consisting of n elements corresponding

to rotations of the polygon, and n corresponding to reflections. In 1983 it was proved by Dragan Marušič [65] that this conjecture is true for abelian groups.

Theorem 2.3.5 [65] *A Cayley graph $\Gamma = Cay(G, S)$ of an abelian group G with at least three vertices contains a Hamiltonian cycle.*

In 1989 Brian Alspach and Cun–Quan Zhang proved that every cubic Cayley graph of a dihedral group is Hamiltonian [2].

A rare positive result for all finite groups was obtained in 2009 by Pak and Radoičić [68].

Theorem 2.3.6 [68] *Every finite group G of size $|G| \geqslant 3$ has a generating set S of size $|S| \leqslant \log_2 |G|$ such that the corresponding Cayley graph $\Gamma = Cay(G, S)$ has a Hamiltonian cycle.*

This theorem shows that every finite group G has a Hamiltonian Cayley graph with a generating set of small size. The bound on S is reached on the group $G = \mathbb{Z}_2^n$ for which the size of its smallest generating set is equal to $\log_2 |G|$. For other groups the size of a generating set is much smaller. For example, for all finite simple groups it is equal to two. This result can be also considered as a corollary of the following natural conjecture.

Conjecture 2.3.7 [68] *There exists $\varepsilon > 0$, such that for every finite group G and every $k \geqslant \varepsilon \log_2 |G|$, the probability $P(G, k)$ that the Cayley graph $\Gamma = Cay(G, S)$ with a random generating set S of size k contains a Hamiltonian cycle, satisfies $P(G, k) \to 1$ as $|G| \to \infty$.*

On one hand, this conjecture is much weaker then the Lovász conjecture. On the other hand, it also does not contradict the Babai conjecture. A work by Michael Krivelevich and Benny Sudakov [57] shows that for every $\varepsilon > 0$ a Cayley graph $\Gamma = Cay(G, S)$ with large enough $|G|$, formed by choosing a set S of $\varepsilon(\log_2 |G|)^5$ random generators in a group G, is almost surely Hamiltonian. Thus, they reduce the bound in Conjecture 2.3.7 down to $k \geqslant \varepsilon(\log_2 |G|)^5$.

We present the proof of Theorem 2.3.6 in the next Section.

2.4 Hamiltonicity of Cayley graphs on finite groups

Let G be a finite group of order n and $H \subset G$ be a subgroup of G. Then for $g \in G$ the sets $gH = \{gh \mid h \in H\}$ and $Hg = \{hg \mid h \in H\}$ are left and right cosets of H in G. A subgroup H of a group G is called a *normal subgroup* ($H \lhd G$) if the sets of left and right cosets of this subgroup in G coincide, i.e. $gH = Hg$ for any $g \in G$. A *simple group* is a nontrivial group whose only normal subgroups are the trivial group and the group itself.

A factor group G/H of a group G with a normal subgroup H is called the set of all cosets of H such that $(aH)(bH) = (ab)H$. A *composition series* of a group G is a subnormal series such that

$$e = H_0 \lhd H_1 \lhd \ldots \lhd H_n = G,$$

with strict inclusions, where H_i is a maximal normal subgroup of H_{i+1} for all $0 \leqslant i \leqslant n$. Equivalently, a composition series is a subnormal series such that each factor group H_{i+1}/H_i is simple. The factor groups are called *composition factors*.

We need the following simple "reduction lemma".

Lemma 2.4.1 *Let G be a finite group and let $H \lhd G$ be a normal subgroup. Suppose $S = S_1 \cup S_2$ is a generating set of G such that $S_1 \subset H$ generate H, and projection S_2' of S_2 onto G/H generates G/H. Suppose both $\Gamma_1 = Cay(H, S_1)$ and $\Gamma_2 = Cay(G/H, S_2')$ contain Hamiltonian paths. Then $\Gamma = Cay(G, S)$ also contains a Hamiltonian path.*

Proof. Let $\Gamma = Cay(G, S)$ be a Cayley graph which contains a Hamiltonian path. By vertex–transitivity of Γ one can arrange this path to start at any vertex $g \in G$.

Let $k = |G/H|$ and let $g_1 = e \in G$. Consider a Hamiltonian path in the Cayley graph $\Gamma_2 = Cay(G/H, S_2')$:

$$H \to Hg_1 \to Hg_2 \to Hg_3 \to \ldots \to Hg_k.$$

Now proceed by induction in a manner similar to that in the proof of Theorem 2.2.1. Fix a Hamiltonian path in the coset Hg_1 so that $e \in G$ is its starting point. Suppose h_1g_1 is its endpoint. Add an edge $\{h_1g_1, h_1g_2\} \in \Gamma$ and consider a Hamiltonian path in the coset Hg_2 starting at h_1g_2. Suppose

h_2g_2 is its endpoint. Repeat until the resulting path ends at h_kg_k. This complete the construction and proves the Lemma. □

Let $l(G)$ be the number of composition factors of G. Denote $r(G)$ and $m(G)$ the number of abelian and non–abelian composition factors, respectively. Clearly, $l(G) = r(G) + m(G)$.

Theorem 2.4.2 *Let G be a finite group and $r(G)$, $m(G)$ be as above. Then there exists a generating set S of G with $|S| \leqslant r(G)+2m(G)$ such that the corresponding Cayley graph $\Gamma = Cay(G, S)$ contains a Hamiltonian path.*

Proof. It is a well known corollary from the classification of finite simple group, that every non–abelian finite simple group can be generated by two elements, one of which is an involution. So, Theorem 2.2.2 is applicable, and every non–abelian finite simple group produces a generating set S, $|S| = 2$, such that the corresponding Cayley graph contains a Hamiltonian cycle. If the group G is cyclic, a single generators suffices, of course.

Now we use Lemma 2.4.1. Observe that in notation Lemma 2.4.1, any generating set S_2' of G/H can be lifted to $S_2 \subset G$, so that $S = S_1 \cup S_2$ is a generating set of G. Therefore, if H and G/H have generating sets of size k_1 and k_2, respectively, so that the corresponding Cayley graphs contain Hamiltonian paths, then G contains such a generating set of size $k_1 + k_2$.

Now fix any composition series of a finite group G. By Lemma 2.4.1, we can construct a generating set of size $r(G) + 2m(G)$, so that the corresponding Cayley graph $\Gamma = Cay(G, S)$ has a Hamiltonian path. This completes the proof of Theorem. □

Now we are ready to prove Theorem 2.3.6.

Proof of Theorem 2.3.6

Proof. We deduce it from Theorem 2.4.2. Fix a composition series of G. Let $r = r(G)$ and $m = m(G)$. Denote by $K_1, \ldots, K_r$ and $L_1, \ldots, L_m$ the abelian and non–abelian composition factors of G, respectively. Since the smallest simple non–abelian group A_5 has order 60, then $|L_j| \geqslant 60 > 4$ for any $j \in \{1, \ldots, m\}$. We have:

$$2^{r+2m} = 2^r \cdot 4^m \leqslant \prod_{i=1}^{r} |K_i| \cdot \prod_{j=1}^{m} |L_i| = |G|.$$

Therefore, $r(G) + 2m(G) \leqslant log_2|G|$, with the equality attained only for $G \cong \mathbb{Z}_2^n$. In the latter case, when $n \geqslant 2$, an elementary inductive argument (or a Gray code (see Section 2.1)) gives a Hamiltonian cycle. In other cases, one can add to a generating set, one extra element, which connects the endpoints of a Hamiltonian path. This gives the desired Hamiltonian cycle and completes the proof. □

2.5 Hamiltonicity of Cayley graphs on the symmetric group

Some results for Cayley graphs on the symmetric group Sym_n generated by transpositions are known. These graphs have been proposed as models for the design and analysis of interconnection networks (see [39, 59]). Moreover, Hamiltonian paths in Cayley graphs on Sym_n provide an algorithm for creating the elements of Sym_n from a particular generating set. The following result was proved by Vladimir Kompel'makher and Vladimir Liskovets [48] in 1975.

Theorem 2.5.1 [48] *The graph $Cay(Sym_n, S)$ is Hamiltonian whenever S is a generating set for Sym_n consisting of transpositions.*

This result was generalized by Tchuente [78] in 1982 as follows.

Theorem 2.5.2 [78] *Let S be a generating set of transpositions for Sym_n. Then there is a Hamiltonian path in the graph $Cay(Sym_n, S)$ joining any permutations of opposite parity.*

Thus, by these statements Cayley graphs on the symmetric group Sym_n, generated by any sets of transpositions, are Hamiltonian. Independently, a number of results were shown for particular sets of generators based on transpositions. In 1991 it was shown by J.–S. Jwo et al. [42] that the Star graph $Sym_n(st)$ is Hamiltonian, and by Jwo [41] that the Bubble–sort graph $Sym_n(t)$ is Hamiltonian. Hamiltonian properties of a Cayley graph generated by transpositions $(l2), (l \cdots n), (n \cdots l)$ were considered in 1993 by R. C. Compton and S. G. Williamson [24]. They defined a doubly adjacent Gray code for the symmetric group Sym_n, gave a procedure for

constructing such a Gray code and showed that this code correspond to a Hamiltonian cycle in the corresponding Cayley graph for $n \geqslant 3$.

Hamiltonicity of the Pancake graph P_n has been investigated independently by Shmuel Zaks [81] in 1984, by Jwo [41] in 1991, by Arkady Kanevsky and Chao Feng [43] in 1995, by Jyh–Jian Sheu et al. [76] in 1999. As for a work of Zaks, he didn't consider the Pancake graph but he presented a new algorithm for generating permutations which exactly gives a Hamiltonian cycle in this graph.

Theorem 2.5.3 *The Pancake graph P_n is Hamiltonian for any $n \geqslant 3$.*

In the next section we present two algorithms to generate a Hamiltonian cycle in the Pancake graph.

2.6 Hamiltonicity of the Pancake graph

Let us recall that the Pancake graph P_n, $n \geqslant 2$, is the Cayley graph on the symmetric group Sym_n of permutations $\pi = [\pi_1\, \pi_2\, \ldots\, \pi_n]$, where $\pi_i = \pi(i)$, $1 \leqslant i \leqslant n$, with the generating set $PR = \{r_i \in Sym_n : 2 \leqslant i \leqslant n\}$ of all prefix–reversals r_i reversing the order of any substring $[1, i]$, $2 \leqslant i \leqslant n$, of a permutation π when multiplied on the right, i.e.

$$[\pi_1\, \ldots\, \pi_i\, \pi_{i+1}\, \ldots\, \pi_n]\, r_i = [\pi_i\, \ldots\, \pi_1\, \pi_{i+1}\, \ldots\, \pi_n].$$

It is a connected vertex–transitive $(n-1)$–regular graph of order $n!$.

Moreover, the Pancake graph P_n, $n \geqslant 3$, has a hierarchical structure such that for any $n \geqslant 3$ it consists of n copies $P_{n-1}(i)$, $1 \leqslant i \leqslant n$, where the vertex set is presented by all permutations with a fixed element at the last position:

$$V^i = \{[\pi_1 \ldots \pi_{n-1} i],\ \text{where } \pi_k \in \{1, \ldots, n\} \backslash \{i\} : 1 \leqslant k \leqslant n-1\},$$

where $|V^i| = (n-1)!$, and the edge set is presented by the set:

$$E_i = \{\{[\pi_1\, \ldots\, \pi_{n-1}\, i], [\pi_1\, \ldots\, \pi_{n-1}\, i]\, r_j\} : 2 \leqslant j \leqslant n-1\},$$

where $|E_i| = \frac{(n-1)!(n-2)}{2}$.

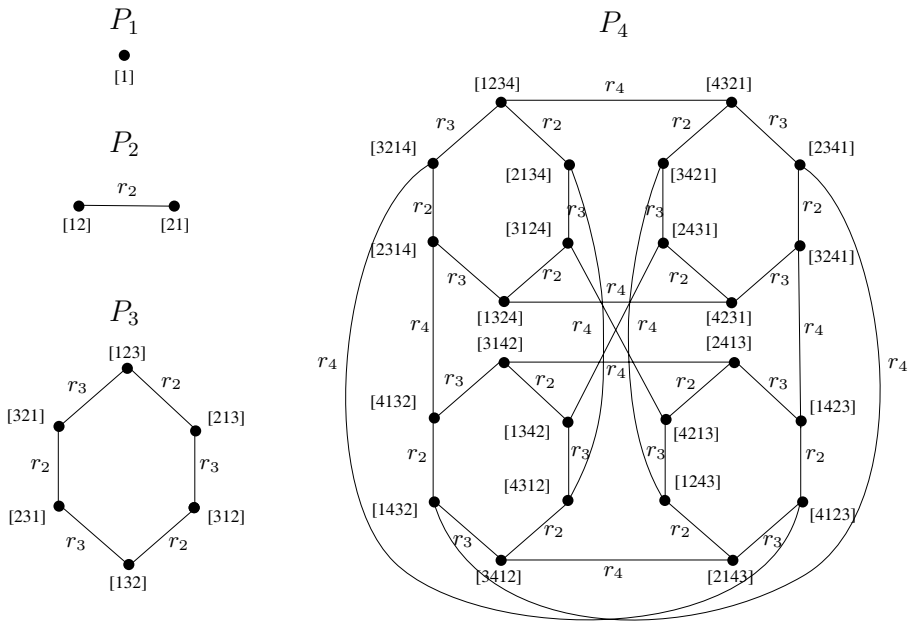


Figure 11. The hierarchical structure of P_2, P_3 and P_4.

There are $(n-2)!$ *external edges* between any two copies $P_{n-1}(i)$ and $P_{n-1}(j)$, $i \neq j$, presented as $\{[i\,\pi_2\,\ldots\,\pi_{n-1}\,j],\,[j\,\pi_{n-1}\,\ldots\,\pi_2,i]\}$, where

$$[i\,\pi_2\,\ldots\,\pi_{n-1}\,j]\,r_n = [j\,\pi_{n-1}\,\ldots\,\pi_2\,i].$$

As one can see, these edges are defined by the prefix–reversal r_n. Prefix–reversals $r_j,\ 2 \leqslant j \leqslant n-1$, define *internal edges* in all n copies $P_{n-1}(i)$, $1 \leqslant i \leqslant n$. Copies $P_{n-1}(i)$, or just P_{n-1} when it is not important to specify the last element of permutations belonging to the copy, are also called $(n-1)$*–copies*. Figure 11 shows the hierarchical structure of P_2, P_3, P_4.

2.6.1 Hamiltonicity based on hierarchical structure

We shall construct a Hamiltonian cycle in the Pancake graph by induction on the size k of the graph P_k, $k \geqslant 3$, with respect to the proof given in [76]. A similar approach was also used in [43].

When $k = 3$ then $P_k \cong C_6$, hence it is Hamiltonian with the following cycle representaion:

$$[123] \overset{r_2}{\to} [213] \overset{r_3}{\to} [312] \overset{r_2}{\to} [132] \overset{r_3}{\to} [231] \overset{r_2}{\to} [321] \overset{r_3}{\to} [123].$$

It is obvious that by removing one edge from this cycle we get a Hamiltonian path.

Now we suppose that we have a Hamiltonian cycle for $k = n - 1$. Let us show that this is also true for $k = n$.

We construct a Hamiltonian cycle H_n by using the hierarchical structure of the graph. Since P_n is a vertex–transitive then without loss of generality one can start with a vertex $\pi_1 = I_n = [12 \dots (n-1)\, n] \in P_{n-1}(n)$.

By inductive assumption there is a Hamiltonian cycle H^n_{n-1} in a copy $P_{n-1}(n)$. We remove an edge $\{[12 \dots (n-1)\, n], [(n-1) \dots 21\, n]\}$ from H^n_{n-1} and denote a Hamiltonian path as L^n_{n-1}, where n represents a corresponding $(n-1)$–copy. A vertex $\pi_2 = [(n-1) \dots 21\, n]$ in a copy $P_{n-1}(n)$ is connected to a vertex $\pi_3 = [n\, 12 \dots (n-2)\, (n-1)]$ by an external edge. Moreover, by our inductive assumption there exists a Hamiltonian cycle in a copy $P_{n-1}(n-1)$. Then we remove an edge $\{[n\, 12 \dots (n-2)\, (n-1)], [(n-2) \dots 1\, n\, (n-1)]\}$ from this Hamiltonian cycle and obtain a Hamiltonian path L^{n-1}_{n-1}. Again, a vertex $\pi_4 = [(n-2) \dots 1\, n\, (n-1)]$ is connected by an external edge to a vertex $\pi_5 = [(n-1)\, n\, 1 \dots (n-3)\, (n-2)]$ of a copy $P_{n-1}(n-2)$ which also has, by inductive assumption, a Hamiltonian cycle. Constructing a Hamiltonian path in this manner for all copies $P_{n-1}(j)$, $1 \leqslant j \leqslant n-2$, finally we have paths $L^{n-2}_{n-1}, \dots, L^1_{n-1}$ that are connected to each other in sequence by an external edge. The last path L^1_{n-1} ends a vertex $\pi_{2n} = [n\, (n-1) \dots 21]$, connected with a vertex π_1 of a copy $P_{n-1}(n)$. Let us note that we started our construction with this copy.

Thus, by combining all these paths $L^n_{n-1}, L^{n-1}_{n-1}, \dots, L^1_{n-1}$ with all external edges between them, finally we obtain a Hamiltonian cycle H_n. This completes a construction.

Figure 12 shows the way to construct a Hamiltonian cycle by the construction above.

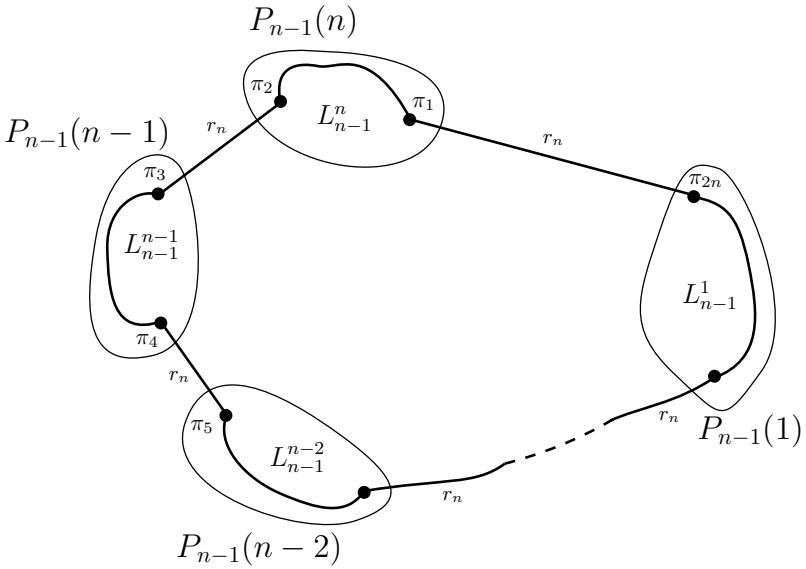


Figure 12. A Hamiltonian cycle of the Pancake graph

A different way to construct a Hamiltonian cycle was considered by Zaks [81] when he generated the permutation in some special order. In his algorithm each successive permutation is generated by reversing a suffix of the preceding permutation. From symmetrical point of view, it is the same as consider a prefix of a permutation that is used in the Pancake graph.

2.6.2 The generating algorithm by Zaks

We start with the identity permutation $I_n = [12\ldots n]$ and in each step reverse a certain suffix. The sequence of sizes of these suffixes is denoted by ζ_n and is defined by recursively as follows (a sequence is written as a concatenation of its elements):

$$\zeta_2 = 2$$
$$\zeta_n = (\zeta_{n-1}\, n)^{n-1}\, \zeta_{n-1},\ n > 2.$$

For example, if $n = 2$ then $\zeta_2 = 2$ and we have:

$$[\underline{12}]\quad [21]$$

If $n = 3$ then $\zeta_3 = 23232$ and we have:

$$[1\underline{23}] \quad [2\underline{31}] \quad [3\underline{12}]$$
$$[\underline{132}] \quad [\underline{213}] \quad [321]$$

If $n = 4$ then $\zeta_4 = 232324232324232324232$ and we have:

$$[12\underline{34}] \quad [23\underline{41}] \quad [34\underline{12}] \quad [41\underline{23}]$$
$$[1\underline{243}] \quad [2\underline{314}] \quad [3\underline{421}] \quad [4\underline{132}]$$
$$[13\underline{42}] \quad [24\underline{13}] \quad [31\underline{24}] \quad [42\underline{31}]$$
$$[1\underline{324}] \quad [2\underline{431}] \quad [3\underline{142}] \quad [4\underline{213}]$$
$$[14\underline{23}] \quad [21\underline{34}] \quad [32\underline{41}] \quad [43\underline{12}]$$
$$[\underline{1432}] \quad [\underline{2143}] \quad [\underline{3214}] \quad [4321]$$

We prove validity of this generating algorithm by induction on n that, starting with the permutation $[12\ldots n]$ and applying the sequence ζ_n of suffix reversals, we generate all $n!$ permutations ending with the permutation $[n\,n-1\ldots1]$.

The assertion holds for $n = 2$. Assuming it holds for $n-1$, we prove it for n: ζ_n starts with ζ_{n-1}. Hence, starting with $[12\ldots n]$ we first generate the $(n-1)!$ permutations that start with 1 and, by the induction assumption, the last one is $[1n\,n-1\ldots2]$. The next element in ζ_n is n, hence the next permutation generated is $[23\ldots n\,1]$. The following elements thus generated are all permutations starting with 2. Continuing in this manner, we then generate the permutations starting with $3\ldots n$.

Moreover, because the first permutation starting with 2 ($[23\ldots n\,1]$) is obtained from the first permutation starting with 1 ($[12\ldots n]$) by increasing each element by 1 (while n becomes 1), and because both the permutations starting with 1 and whose starting with 2 are generated by the sequence ζ_{n-1}, therefore the last permutation starting with 2 is derived from $[1n\,n-1\ldots2]$ (the last permutation starting with 1) in the same manner, namely, it is $[21\,n\ldots3]$. Continuing in this manner it is easy to show that the first permutation starting with $i, 1 < i \leqslant n$, is $[i\,i+1\ldots n\,12\ldots i-1]$ and the last one is $[i\,i-1\ldots1\,n\,n-1\ldots i+1]$ (it is $[n\,n-1\ldots1]$ for $i = n$). The proof is thus complete. □

2.7 Other cycles of the Pancake graph

It is also known that the Pancake graph P_n, $n \geqslant 3$, contains many other cycles. In 1995 it was shown by Arkady Kanevsky and Chao Feng [43] that all cycles of length l, where $6 \leqslant l \leqslant n! - 2$ and $l = n!$ can be embedded in the Pancake graph P_n, $n \geqslant 3$. In 1999 it was shown by Jyh–Jian Sheu etc. [76] that a cycle of length $l = n! - 1$ can be also embedded in the Pancake graph P_n, $n \geqslant 3$. So, finally the following result takes place.

Theorem 2.7.1 [43, 76] *All cycles of length l, where $6 \leqslant l \leqslant n!$, can be embedded in the Pancake graph P_n, $n \geqslant 3$, but there are no cycles of length $3, 4,$ or 5.*

An explicit description of cycles is gradually being developed. The first results concerning cycle characterization in the Pancake graph were obtained in [53] where the following cycle representation via a product of generating elements was used.

A sequence of prefix–reversals $C_l = r_{i_0} \dots r_{i_{l-1}}$, where $2 \leqslant i_j \leqslant n$, and $i_j \neq i_{j+1}$ for any $0 \leqslant j \leqslant l-1$, such that $\pi r_{i_0} \dots r_{i_{l-1}} = \pi$, where $\pi \in Sym_n$, is called *a form of a cycle C_l of length l*. A cycle C_l of length l is also called an *l–cycle*, and a vertex of P_n is identified with the permutation which corresponds to this vertex. It is evident that any l–cycle can be presented by $2l$ forms (not necessarily different) with respect to a vertex and a direction. *The canonical form C_l of an l–cycle* is called a form with a lexicographically maximal sequence of indices $i_0 \dots i_{l-1}$. Two cycles in a graph are *independent* if they do not have any vertex in common.

In this Section we present results on the full characterization of cycles of length 6, 7 and 8 in the Pancake graph. The proof for 9–cycles one could be found in [54].

The main results of this Section are presented by the following theorems.

Theorem 2.7.2 [53] *The Pancake graph P_n, $n \geqslant 3$, has $\frac{n!}{6}$ independent 6–cycles of the canonical form*

$$C_6 = r_3\, r_2\, r_3\, r_2\, r_3\, r_2. \tag{2.2}$$

Moreover, each of the vertices of P_n belongs to exactly one 6–cycle.

Theorem 2.7.3 [53] *The Pancake graph P_n, $n \geqslant 4$, has $n!\,(n-3)$ distinct 7–cycles of the canonical form*

$$C_7 = r_k\, r_{k-1}\, r_k\, r_{k-1}\, r_{k-2}\, r_k\, r_2, \tag{2.3}$$

where $4 \leqslant k \leqslant n$. Moreover, each of the vertices of P_n belongs to $7\,(n-3)$ distinct 7–cycles and there are $\frac{n!}{8} \leqslant n_7 \leqslant \frac{n!}{7}$ independent 7–cycles.

As one can see, the descriptions of 6–cycles and 7–cycles are not so complicated. However, the situation is changed dramatically for 8–cycles.

Theorem 2.7.4 [55] *Each of vertices of the Pancake graph P_n, $n \geqslant 4$, belongs to N_8 distinct 8–cycles of the following canonical forms:*

$$C_8^1 = r_k\, r_j\, r_i\, r_j\, r_k\, r_{k-j+i}\, r_i\, r_{k-j+i}, \qquad 2 \leqslant i < j \leqslant k-1,\ 4 \leqslant k \leqslant n, \tag{2.4}$$

$$C_8^2 = r_k\, r_{k-1}\, r_2\, r_{k-1}\, r_k\, r_2\, r_3\, r_2, \qquad 4 \leqslant k \leqslant n, \tag{2.5}$$

$$C_8^3 = r_k\, r_{k-i}\, r_{k-1}\, r_i\, r_k\, r_{k-i}\, r_{k-1}\, r_i, \qquad 2 \leqslant i \leqslant k-2,\ 4 \leqslant k \leqslant n, \tag{2.6}$$

$$C_8^4 = r_k\, r_{k-i+1}\, r_k\, r_i\, r_k\, r_{k-i}\, r_{k-1}\, r_{i-1}, \qquad 3 \leqslant i \leqslant k-2,\ 5 \leqslant k \leqslant n, \tag{2.7}$$

$$C_8^5 = r_k\, r_{k-1}\, r_{i-1}\, r_k\, r_{k-i+1}\, r_{k-i}\, r_k\, r_i, \qquad 3 \leqslant i \leqslant k-2,\ 5 \leqslant k \leqslant n, \tag{2.8}$$

$$C_8^6 = r_k\, r_{k-1}\, r_k\, r_{k-i}\, r_{k-i-1}\, r_k\, r_i\, r_{i+1}, \qquad 2 \leqslant i \leqslant k-3,\ 5 \leqslant k \leqslant n, \tag{2.9}$$

$$C_8^7 = r_k\, r_{k-j+1}\, r_k\, r_i\, r_k\, r_{k-j+1}\, r_k\, r_i, \qquad 2 \leqslant i < j \leqslant k-1,\ 4 \leqslant k \leqslant n, \tag{2.10}$$

$$C_8^8 = r_4\, r_3\, r_4\, r_3\, r_4\, r_3\, r_4\, r_3, \tag{2.11}$$

where $N_8 = \frac{n^3+12n^2-103n+176}{2}$. Moreover, there are $\frac{n!N_8}{8}$ distinct 8–cycles and a maximal set of independent 8–cycles will contain $\frac{n!}{8}$ of these.

Proofs of these results are based on the hierarchical structure of the Pancake graph. We also need some new definitions and notations.

A *segment* $[\pi_i \dots \pi_j]$ of a permutation $\pi = [\pi_1 \dots \pi_i \dots \pi_j \dots \pi_n]$ consists of all elements that lie between π_i and π_j inclusive. Any permutation can be written as a sequence of singleton and multiple segments. We use characters from $\{i, j, k\}$ to denote singletons and characters from $\{\alpha, \beta, \gamma\}$ to denote multiple segments. For example, $\pi = [i\, \pi_2\, \pi_3\, \pi_4\, j\, \pi_6\, \pi_7\, \pi_8\, k]$ can be presented as $\pi = [i\, \alpha\, j\, \beta\, k]$ where $\alpha = [\pi_2\, \pi_3\, \pi_4]$, $\beta = [\pi_6\, \pi_7\, \pi_8]$. If $\overline{\alpha}$ is

the inversion of a segment α then $\overline{\overline{\alpha}} = \alpha$. Let us denote the number of elements in a segment α as $|\alpha|$. We also put $\overline{\pi} = \pi\, r_n$ and $\overline{\tau} = \tau\, r_n$. Note we observe empty segments where it does not contradict initial definitions.

Lemma 2.7.5 [53] *Let two distinct permutations π and τ belong to the same $(n-1)$–copy of P_n, $n \geqslant 3$, and let $d(\pi, \tau) \leqslant 2$, then $\overline{\pi}$, $\overline{\tau}$ belong to distinct $(n-1)$–copies of the graph.*

Proof. Let $\pi, \tau \in P_{n-1}(i)$, $1 \leqslant i \leqslant n$. If $d(\pi, \tau) = 1$ and if we put $\pi = [j\, \alpha\, k\, \beta\, i]$ then $\tau = [k\, \overline{\alpha}\, j\, \beta\, i]$ where $j \neq k \neq i$. So, $\overline{\pi} = [i\, \overline{\beta}\, k\, \overline{\alpha}\, j]$, $\overline{\tau} = [i\, \overline{\beta}\, j\, \alpha\, k]$, which means that $\overline{\pi}$, $\overline{\tau}$ belong to the distinct copies $P_{n-1}(j)$ and $P_{n-1}(k)$. If $d(\pi, \tau) = 2$ then there is a permutation ω in $P_{n-1}(i)$ adjacent to π and τ. Permutations π and τ are obtained from ω by multiplying on different (not equal to r_n) prefix–reversals on the right. Thereby, the first elements of π and τ should be different hence $\overline{\pi} = \pi\, r_n$ and $\overline{\tau} = \tau\, r_n$ should be different, i.e. they belong to the distinct $(n-1)$–copies of P_n. □

2.7.1 6–cycles of the Pancake graph

In this section we present the proof of Theorem 2.7.2.

Proof. If $n = 3$ then $P_3 \cong C_6$ and there is the only 6–cycle presented as

$$[123] \overset{r_2}{\to} [213] \overset{r_3}{\to} [312] \overset{r_2}{\to} [132] \overset{r_3}{\to} [231] \overset{r_2}{\to} [321] \overset{r_3}{\to} [123]$$

for which the canonical form is $C_6 = r_3\, r_2\, r_3\, r_2\, r_3\, r_2$.

Let us show that there are no other forms of 6–cycles in P_n, $n \geqslant 4$. First of all, we prove that an 6–cycle doesn't appear on vertices of two distinct $(n-1)$–copies. Indeed, if $\pi, \tau \in P_{n-1}(i)$ and $\overline{\pi}, \overline{\tau} \in P_{n-1}(j)$ then $d(\pi, \tau) \neq 1$ and $d(\pi, \tau) \neq 2$ by Lemma 2.7.5, and hence $d(\pi, \tau) \geqslant 3$. Suppose that there is an 6–cycle containing vertices $\pi, \tau, \overline{\pi}, \overline{\tau}$. So, if $d(\pi, \tau) = 3$ then $\overline{\pi}$, $\overline{\tau}$ are adjacent in $P_{n-1}(j)$ and by Lemma 2.7.5 vertices $\pi = \overline{\overline{\pi}}\, r_n, \tau = \overline{\overline{\tau}}\, r_n$ belong to the distinct $(n-1)$–copies but this is not true since $\pi, \tau \in P_{n-1}(i)$. If $d(\pi, \tau) = 4$ then $\overline{\pi} = \overline{\tau}$ but this is not possible since $\pi \neq \tau$. Thus, an 6–cycle doesn't appear on vertices of two distinct $(n-1)$–copies.

Now let us show that an 6–cycle doesn't appear on vertices of three distinct $(n-1)$–copies. Let $\pi, \tau \in P_{n-1}(i)$ such that $d(\pi, \tau) \leqslant 2$ then

by Lemma 2.7.5 vertices $\overline{\pi}$, $\overline{\tau}$ belong to the distinct $(n-1)$–copies. We consider two cases.

Case 1. If $d(\pi,\tau)=1$ then vertices $\pi,\tau,\overline{\pi},\overline{\tau}$ might be belong to an 6–cycle if and only if $d(\overline{\pi},\overline{\tau})=3$. Show that this is not true. Let $\pi=[j\,\alpha\,k\,\beta\,i]$ then $\tau=[k\,\overline{\alpha}\,j\,\beta\,i]$ and $\overline{\pi}=[i\,\overline{\beta}\,k\,\overline{\alpha}\,j]\in P_{n-1}(j)$, $\overline{\tau}=[i\,\overline{\beta}\,j\,\alpha\,k]\in P_{n-1}(k)$. The shortest path starting at $\overline{\pi}$ and belonging to $P_{n-1}(k)$ should contain vertices $\omega=[k\,\beta\,i\,\overline{\alpha}\,j]$ and $\overline{\omega}=[j\,\alpha\,i\,\overline{\beta}\,k]\in P_{n-1}(k)$, i.e. $d(\overline{\pi},\overline{\omega})=2$. It is evident that there is no a prefix–reversal transforming $\overline{\omega}$ into $\overline{\tau}$, i.e. $d(\overline{\omega},\overline{\tau})\neq 1$, and hence $d(\overline{\pi},\overline{\tau})\neq 3$.

Case 2. If $d(\pi,\tau)=2$ then vertices $\pi,\tau,\overline{\pi},\overline{\tau}$ might be belong to an 6–cycle if and only if $d(\overline{\pi},\overline{\tau})=2$. However this is not possible since by Lemma 2.7.5 vertices $\pi=\overline{\pi}\,r_n$ and $\tau=\overline{\tau}\,r_n$ belong to the distinct $(n-1)$–copies. Thus, an 6–cycle doesn't appear on vertices of three distinct $(n-1)$–copies.

It is also evident that an 6–cycle doesn't appear on vertices of four and more distinct $(n-1)$–copies since there should be at least four external edges and at least one edge in each of $(n-1)$–copies, so we have an 8–cycle.

Thus, there is the only canonical form, namely $r_3\,r_2\,r_3\,r_2\,r_3\,r_2$, to describe 6–cycles in P_n, $n\geqslant 3$. These cycles are independent for $n\geqslant 4$ since prefix–reversals r_i, $4\leqslant i\leqslant n$, define external edges for 6–cycles, which means that each of the vertices of P_n belongs to the only 6–cycle. □

2.7.2 7–cycles of the Pancake graph

Proof. We prove Theorem 2.7.3 by the induction on the dimension k of the Pancake graph P_k when $k\geqslant 4$.

If $k=3$ then there are no 7–cycles in $P_3\cong C_6$.

If $k=4$ then theorem says that each of vertices of P_4 belongs to 7 distinct 7–cycles. Since P_n is a vertex–transitive graph then it is enough to check this fact for any its vertex. In particular, all 7–cycles containing the identity permutation $I=[1234]$ are presented in the Table 1. They could be found easily by considering vertex distributions of P_4 in all metric spheres centered at the identity permutation. The canonical form for all cycles presented in Table 1 is $C_7=r_4\,r_3\,r_4\,r_3\,r_2\,r_4\,r_2$ that corresponds to (2.3) when $k=4$.

	vertex description	prefix–reversal description
1	[1234]-[4321]-[2341]-[1432]-[3412]-[4312]-[2134]	$r_4\, r_3\, r_4\, r_3\, r_2\, r_4\, r_2$
2	[1234]-[3214]-[4123]-[2143]-[1243]-[3421]-[4321]	$r_3\, r_4\, r_3\, r_2\, r_4\, r_2\, r_4$
3	[1234]-[4321]-[2341]-[3241]-[1423]-[4123]-[3214]	$r_4\, r_3\, r_2\, r_4\, r_2\, r_4\, r_3$
4	[1234]-[3214]-[2314]-[4132]-[1432]-[2341]-[4321]	$r_3\, r_2\, r_4\, r_2\, r_4\, r_3\, r_4$
5	[1234]-[2134]-[4312]-[3412]-[2143]-[4123]-[3214]	$r_2\, r_4\, r_2\, r_4\, r_3\, r_4\, r_3$
6	[1234]-[4321]-[3421]-[1243]-[4213]-[3124]-[2134]	$r_4\, r_2\, r_4\, r_3\, r_4\, r_3\, r_2$
7	[1234]-[2134]-[4312]-[1342]-[2431]-[3421]-[4321]	$r_2\, r_4\, r_3\, r_4\, r_3\, r_2\, r_4$

Table 1. 7–cycles in P_4 containing the identity permutation $I = [1234]$.

Now we assume that theorem is hold for $k = n-1$ and prove that it is hold also for $k = n$ using the hierarchical structure of P_n.

By the induction assumption, any vertex of any $(n-1)$–copy belongs to $7((n-1)-3) = 7(n-4)$ distinct 7–cycles of this copy. However, besides 7–cycles belonging to the same $(n-1)$–copy there may also be 7–cycles belonging to the distinct $(n-1)$–copies of the graph. The following three cases are possible.

Case 1. Suppose that such an 7–cycle C_7^* is formed on vertices from two copies $P_{n-1}(i)$ and $P_{n-1}(j)$, $1 \leqslant i \neq j \leqslant n$, such that either two vertices of C_7^* belong to $P_{n-1}(i)$ and other five vertices belong to $P_{n-1}(j)$, or three vertices of C_7^* belong to $P_{n-1}(i)$ and other four vertices belong to a copy $P_{n-1}(j)$. In the both cases we have $d(\pi, \tau) \leqslant 2$ for any vertices π, $\tau \in P_{n-1}(i)$ belonging to C_7^*. Then by Lemma 2.7.5 vertices $\overline{\pi}$, $\overline{\tau}$ belong to the distinct $(n-1)$–copies that contradicts to our assumption. Therefore, an 7–cycle does not occur in this case.

Case 2. Suppose that such an 7–cycle C_7^* is formed on vertices from three distinct $(n-1)$–copies such that two vertices π^{i_1}, π^{i_2} belong to $P_{n-1}(i)$, two vertices π^{j_1}, π^{j_2} belong to $P_{n-1}(j)$, the other three vertices π^{n_1}, π^{n_2}, π^{n_3} belong to $P_{n-1}(n)$, where $1 \leqslant i < j \leqslant n$ (see Figure 13).

Let us describe such a cycle. Since P_n is a vertex–transitive graph then there is no loss of generality in taking $\pi^{n_2} = I_n = [\alpha\, i \beta\, j\, \gamma\, n]$, where $\alpha = [1 \ldots i-1]$, $\beta = [i+1 \ldots j-1]$, $\gamma = [j+1 \ldots n-1]$ and $|\alpha| = i-1$, $|\beta| = j-i-1$, $|\gamma| = n-j-1$. By Lemma 2.7.5 vertices π^{n_1} and π^{n_3} are adjacent to vertices from distinct $(n-1)$–copies $P_{n-1}(i)$ and $P_{n-1}(j)$,

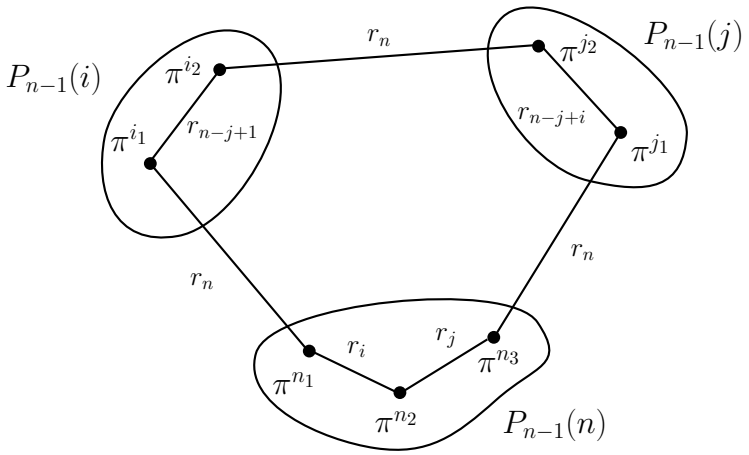


Figure 13. Case 2 for the proof of Theorem 2.7.3

hence these vertices are presented by permutations:

$$\pi^{n_1} = \pi^{n_2} r_i = [i\,\overline{\alpha}\,\beta\,j\,\gamma\,n], \text{ where } \pi^{n_1}_j = j,$$

$$\pi^{n_3} = \pi^{n_2} r_j = [j\,\overline{\beta}\,i\,\overline{\alpha}\,\gamma\,n], \text{ where } \pi^{n_3}_{j-i+1} = i,$$

and their adjacent vertices in $P_{n-1}(i)$ and $P_{n-1}(j)$ are presented by the following permutations:

$$\pi^{i_1} = \pi^{n_1} r_n = [n\,\overline{\gamma}\,j\,\overline{\beta}\,\alpha\,i], \text{ where } \pi^{i_1}_{n-j+1} = j,$$

$$\pi^{j_1} = \pi^{n_3} r_n = [n\,\overline{\gamma}\,\alpha\,i\,\beta\,j], \text{ where } \pi^{j_1}_{n-j+i} = i.$$

A vertex π^{i_2} should be adjacent to the vertex π^{i_1} and to one of the vertices, say π^{j_2}, from $P_{n-1}(j)$. So we have:

$$\pi^{i_2} = \pi^{i_1} r_{n-j+1} = [j\,\gamma\,n\overline{\beta}\,\alpha\,i], \text{ where } \pi^{i_2}_1 = j.$$

On the other hand, a vertex π^{j_2} should be adjacent to the vertex π^{j_1}. Moreover, since it is also adjacent to π^{i_2} hence π^{j_2} has the following form:

$$\pi^{j_2} = \pi^{j_1} r_{n-j+i} = [i\,\overline{\alpha}\,\gamma\,n\,\beta\,j], \text{ where } \pi^{j_2}_1 = i.$$

By our assumption, the vertices π^{i_2} and π^{j_2} are incident to the same external edge, which means that a permutation $\pi^* = \pi^{i_2} r_n = [i\,\overline{\alpha}\,\beta\,n\,\overline{\gamma}\,j]$ should

coincide with the permutation π^{j_2}. This is possible only in the case when segments β and γ are empty, i.e. $|\beta| = j-i-1 = 0$ and $|\gamma| = n-j-1 = 0$. From this we have $j = n-1$ and $i = j-1 = n-2$, and an 7–cycle is presented as follows:

$$\pi^{i_1} \xrightarrow{r_2} \pi^{i_2} \xrightarrow{r_n} \pi^{j_2} \xrightarrow{r_{n-1}} \pi^{j_1} \xrightarrow{r_n} \pi^{n_3} \xrightarrow{r_{n-1}} \pi^{n_2} \xrightarrow{r_{n-2}} \pi^{n_1} \xrightarrow{r_n} \pi^{i_1}.$$

It is evident that its canonical form $C_7 = r_n\, r_{n-1}\, r_n\, r_{n-1}\, r_{n-2}\, r_n\, r_2$ coincides with (2.3) when $k = n$.

Case 3. Suppose that such an 7–cycle is formed on vertices from four or more $(n-1)$–copies. It follows from the hierarchical structure of the graph that any its vertex is incident to the only external edge. So, any 7–cycle in this graph should contain at least two vertices of the same $(n-1)$–copy and hence an 7–cycle does not occur in this assumption.

Thus, the only canonical form $r_n\, r_{n-1}\, r_n\, r_{n-1}\, r_{n-2}\, r_n\, r_2$ representing seven cycles of the length 7 and containing vertices from three distinct $(n-1)$–copies of the graph P_n is found. By vertex–transitivity this is true for any vertex of P_n . By the induction assumption, any vertex of any $(n-1)$–copy belongs to $7(n-4)$ distinct 7–cycles from this copy. Therefore, any vertex of P_n belongs to $7(n-3)$ distinct 7–cycles of the canonical form (2.3) that completes the proof on the main fact of Theorem 2.7.3.

Since each of vertices belongs to $7\,(n-3)$ distinct 7–cycles and there are $n!$ vertices in P_n, hence there are $n!\,7\,(n-3)$ cycles of length 7. However, each cycle was enumerated seven times, so in total there are $n!\,(n-3)$ distinct 7–cycles in P_n.

It is also easy to show that there are at most three independent 7–cycles in P_4. For example, the following three 7–cycles are independent in P_4:

$$C_7^1 = [1234] - [2134] - [4312] - [1342] - [2431] - [3421] - [4321],$$
$$C_7^2 = [3241] - [2341] - [1432] - [3412] - [2143] - [4123] - [1423],$$
$$C_7^3 = [4213] - [2413] - [3142] - [4132] - [2314] - [1324] - [3124].$$

It follows from the hierarchical structure of P_n, $n \geqslant 4$, that there are $\frac{n!}{24}$ copies of P_4 and each of them has exactly three independent 7–cycles. So, in total there are at least $\frac{n!}{8}$ independent 7–cycles that gives the lower bound. The upper bound is obtained in assumption that each of vertices of P_n, $n \geqslant 7$, belongs to the only 7–cycle. $\square$

2.7.3 8–cycles of the Pancake graph

In this section we present the proof of Theorem 2.7.4. For this we need some additional notations and technical lemmas. As before, we suggest that any permutation can be written as a sequence of singleton and multiple segments presented by characters from $\{p, q, s, t\}$ and $\{\alpha, \beta, \gamma, A, B, C\}$ respectively. If $\pi = [\alpha\,\beta]$, where $\alpha = [\pi_1\,\pi_2\,\ldots\,\pi_i]$ and $\beta = [\pi_{i+1}\,\ldots\,\pi_n]$, then $\pi\, r_{|\alpha|} = [\overline{\alpha}\,\beta]$, where $|\alpha|$ is the number of elements in a segment α, $|\alpha| \geqslant 2$, and $\overline{\alpha}$ is the inversion of a segment α. It is obvious that $\overline{\overline{\alpha}} = \alpha$. Note that we allow empty segments where this does not contradict the initial definitions.

An independent set D of vertices in a graph is called an *efficient dominating set* [26] if each vertex not in D is adjacent to exactly one vertex in D. It was shown in [70] that $D_p = \{[p\,\pi_2 \ldots \pi_n] : \pi_j \in \{1, \ldots, n\}\backslash\{p\},\ 2 \leqslant j \leqslant n\}$, $|D_p| = (n-1)!$, $p = 1, \ldots, n$, are efficient dominating sets in P_n, $n \geqslant 3$. Let us note that external edges of P_n, $n \geqslant 3$, are incident to vertices from different efficient dominating sets of the Pancake graph. The *distance* $d = d(\pi, \tau)$ between two vertices $\pi, \tau \in V(P_n)$ is defined as the least number of prefix–reversals transforming π into τ, i.e. $\pi r_{i_1} r_{i_2} \ldots r_{i_d} = \tau$.

The next lemma gives a full list of paths of length three between vertices of the same efficient dominating set.

Lemma 2.7.6 *Two permutations $\pi, \tau \in D_p$, $1 \leqslant p \leqslant n$, are at distance three from each other in P_n, $n \geqslant 3$, if and only if:*
1) either $\tau = \pi\, r_j\, r_i\, r_j$, where $2 \leqslant i < j \leqslant n$, and permutations π, τ are presented as:

$$\pi = [A\,B\,\gamma], \quad \tau = [A\,\overline{B}\,\gamma]; \tag{2.12}$$

2) or $\tau = \pi\, r_j\, r_i\, r_{i-j+1}$, where $2 \leqslant j < i \leqslant n$, and permutations π, τ are presented as:

$$\pi = [p\,A\,B\,\gamma], \quad \tau = [p\,B\,A\,\gamma]. \tag{2.13}$$

Proof. We consider $\pi \in D_p$ such that $\pi = [p\,\alpha\,q\,\beta\,k]$, $\pi_j = q$. Let us find other vertices from D_p being at distance three from π. Let $\pi^1 = \pi\, r_j = [q\,\overline{\alpha}\,p\,\beta\,k]$, where $\pi^1_j = p$, $2 \leqslant j \leqslant n$. An application of a prefix–reversal r_i, $2 \leqslant i \leqslant n$, $i \neq j$, to the permutation π^1 gives us the following two cases.

1) If $i < j$ then $\pi^2 = \pi^1 r_i = [\alpha_2\, q\, \overline{\alpha_1}\, p\, \beta\, k]$, where $\pi^2_j = p$, $\alpha = \alpha_1\, \alpha_2$ and $|\alpha_2| = i - 1$, and an application of a prefix–reversal r_j to the permutation π^2 gives us the permutation:

$$\tau = \pi^2\, r_j = [p\, \alpha_1\, q\, \overline{\alpha_2}\, \beta\, k],$$

hence $\tau = \pi\, r_j\, r_i\, r_j$, and we get (2.12) by setting $A = p\, \alpha_1$, $B = \alpha_2\, q$, $\gamma = \beta\, k$ in π and τ in this case. Note, that using r_j is the only way to restore p to the first position and thus to end at an element of D_p after reaching π^2.

2) If $i > j$ then $\pi^2 = \pi^1\, r_i = [\overline{\beta_1}\, p\, \alpha\, q\, \beta_2\, k]$, where $\pi^2_{i-j+1} = p$, $\beta = \beta_1\beta_2$ and $|\beta_1| = i - j$, and an application of a prefix–reversal r_{i-j+1} to the permutation π^2 gives us the permutation:

$$\tau = \pi^2\, r_{i-j+1} = [p\, \beta_1\, \alpha\, q\, \beta_2\, k],$$

hence $\tau = \pi\, r_j\, r_i\, r_{i-j+1}$, and we get (2.13) setting $A = \alpha\, q$, $B = \beta_1$, $\gamma = \beta_2\, k$ in π and τ in this case. Note, that using r_j is the only way to restore p to the first position and thus to end at an element of D_p after reaching π^2. □

The next lemma gives a description of paths of length three defined on internal edges of the graph between vertices of given forms.

Lemma 2.7.7 *Let permutations π and τ be presented as:*

1) $\pi = [\gamma_1 AB\gamma_2]$ and $\tau = [\gamma_1 \overline{A} B\gamma_2]$. Then:

a) there exists a unique path of length three:

$$\tau = \pi r_{|\gamma_1|+|A|} r_{|A|} r_{|\gamma_1|+|A|}, \tag{2.14}$$

provided that either $|\gamma_1| \geqslant 2$ and $|A| \geqslant 2$, or $|\gamma_1| = 1$ and $|A| \geqslant 3$;

b) there are two paths of length three:

$$\tau = \pi r_2 r_3 r_2, \quad \tau = \pi r_3 r_2 r_3, \tag{2.15}$$

provided that $|\gamma_1| = 1$ and $|A| = 2$;

2) $\pi = [\gamma_1 AB\gamma_2]$ *and* $\tau = [\gamma_1 BA\gamma_2]$*, where* $|\gamma_1| \geqslant 0$*,* $|A| \geqslant 1$*,* $|B| \geqslant 1$*. Then:*

a) there is a unique path of length three:

$$\tau = \pi r_{|\gamma_1|+2} r_2 r_{|\gamma_1|+2}, \tag{2.16}$$

provided that $|\gamma_1| \geqslant 2$*, and* $|A| = |B| = 1$*;*

b) there is a unique path of length three:

$$\tau = \pi r_{|\gamma_1|+|A|} r_{|\gamma_1|+|A|+|B|} r_{|\gamma_1|+|B|}, \tag{2.17}$$

provided that $|\gamma_1| = 1$*, and at least one of* $|A|, |B| \neq 1$*;*

c) there are two paths of length three:

$$\tau = \pi r_2 r_3 r_2 = \pi r_3 r_2 r_3, \tag{2.18}$$

provided that $|\gamma_1| = |A| = |B| = 1$*;*

d) there is a unique path of length three:

$$\tau = \pi r_{|A|} r_{|A|+|B|} r_{|B|}, \tag{2.19}$$

provided that $|\gamma_1| = 0$ *and* $|A| \geqslant 2$*,* $|B| \geqslant 2$*.*
There are no other paths of length three between π *and* τ *of these types.*

Proof. 1) If $\pi = [\gamma_1 AB\gamma_2]$ and $\tau = [\gamma_1 \overline{A} B\gamma_2]$, then (2.14) is checked by a direct verification:

$$[\gamma_1 AB\gamma_2] \overset{r_{|\gamma_1|+|A|}}{\to} [\overline{A}\,\overline{\gamma_1} B\gamma_2] \overset{r_{|A|}}{\to} [A\overline{\gamma_1} B\gamma_2] \overset{r_{|\gamma_1|+|A|}}{\to} [\gamma_1 \overline{A} B\gamma_2].$$

Suppose that there is one more path of length three. Then these two paths should form an 6–cycle. In part (a), either $|\gamma_1| \geqslant 2$ and $|A| \geqslant 2$, or $|\gamma_1| = 1$ and $|A| \geqslant 3$, so $r_{|\gamma_1|+|A|} = r_m$ for some $m \geqslant 4$, but by Theorem 2.7.2, no an 6–cycle includes r_m with $m \geqslant 4$ in its form. Therefore, the given path is the only one in this case. In part (b), $|\gamma_1| = 1$ and $|A| = 2$, so $m = 3$ and

the condition of Theorem 2.7.2 holds, hence we obtain two distinct paths of the stated forms (2.15).

2) If $\pi = [\gamma_1 AB\gamma_2]$ and $\tau = [\gamma_1 BA\gamma_2]$, and $|\gamma_1| \geqslant 2$, $|A| \geqslant 1$, $|B| \geqslant 1$, then there is the following path of length four between these vertices:

$$\pi = [\gamma_1 AB\gamma_2] \overset{r_{|\gamma_1|+|A|}}{\to} [\overline{A}\,\overline{\gamma_1} B\gamma_2] \overset{r_{|\gamma_1|+|A|+|B|}}{\to} [\overline{B}\gamma_1 A\gamma_2] \overset{r_{|\gamma_1|+|B|}}{\to}$$

$$[\overline{\gamma_1} BA\gamma_2] \overset{r_{|\gamma_1|}}{\to} [\gamma_1 BA\gamma_2] = \tau. \tag{2.20}$$

Suppose there is also a path of length three between π and τ. By Theorem 2.7.1, there are no cycles of length 3 or 5, and hence no paths of lengths 3 and 4 exist between two fixed vertices unless the paths are disjoint. This means that these two paths should form an 7–cycle, including the sequence $r_{m+a}r_{m+a+b}r_{m+b}r_m$, where $|\gamma_1| = m$, $|A| = a$ and $|B| = b$. By Theorem 2.7.2, this is possible only in the case when $m = k-2$, $k \geqslant 4$, and $a = b = 1$, which implies that a unique path of length three has the form $r_{m+2}r_2r_{m+2}$ that corresponds to (2.16).

Putting $|\gamma_1| = 1$ in (2.20), a path

$$\tau = \pi r_{|\gamma_1|+|A|} r_{|\gamma_1|+|A|+|B|} r_{|\gamma_1|+|B|},$$

corresponding to (2.17), is obtained. Taking $|\gamma_1| = m$, $|A| = a$ and $|B| = b$, the obtained path is presented as $r_{m+a}r_{m+a+b}r_{m+b}$. Suppose that there is one more path of length three between π and τ. Then these two paths should form an 6–cycle. By Theorem 2.7.2, this is possible only in the case when $m = a = b = 1$, which gives us the paths $\tau = \pi r_2 r_3 r_2$ and $\tau = \pi r_3 r_2 r_3$, corresponding to (2.18).

Putting $|\gamma_1| = 0$ in (2.20), a path

$$\tau = \pi r_{|A|} r_{|A|+|B|} r_{|B|},$$

corresponding to (2.19) with $|A| \geqslant 2$, $|B| \geqslant 2$, is obtained. Suppose there is one more path of length three between π and τ. Then these two paths should form an 6–cycle. By the conditions of Lemma, $|A| + |B| \geqslant 4$, hence $r_{|A|+|B|} = r_m$ for some $m \geqslant 4$, but by Theorem 2.7.2, no an 6–cycle includes r_m with $m \geqslant 4$ in its form. Therefore, the given path is the only one in this case. If $|A| = 1$ or $|B| = 1$, then the path above is transformed into a 2–path or an edge. This completes the proof of the lemma. □

Now we are ready to prove Theorem 2.7.4

Proof of Theorem 2.7.4

Since $P_3 \cong C_6$ and due to the hierarchical structure, P_4 has four copies of P_3, each of which obviously cannot contain 8–cycles. However, P_4 has 8–cycles consisting of paths within copies of P_3 together with external edges between these copies. In general, any 8–cycle of $P_n, n \geqslant 4$, must consist of paths within subgraphs that are isomorphic to P_{k-1} for some $4 \leqslant k \leqslant n$, joined by external edges between these subgraphs. Hence, all 8–cycles of P_n, $n \geqslant 4$, could be found recursively by considering 8–cycles within each P_k, $4 \leqslant k \leqslant n$, consisting of vertices from some copies of P_{k-1}. This approach is used in the proof of Theorem 2.7.4. It is assumed that any copy of P_{k-1} has at least two vertices, since each vertex has a unique external edge. We obtain canonical forms of 8–cycles and count their numbers.

Case 1: an 8–cycle within P_k has vertices from two copies of P_{k-1}

Suppose that an 8–cycle is formed on vertices from copies $P_{k-1}(p)$ and $P_{k-1}(s)$, $1 \leqslant p \neq s \leqslant k$. By Lemma 2.7.5, if two vertices π and τ, belonging the same $(k-1)$–copy, are at the distance at most two, then their external neighbours $\overline{\pi}$ and $\overline{\tau}$ should belong to distinct $(k-1)$–copies. Hence, an 8–cycle cannot occur in situations when its two (three) vertices belong to one copy and six (five) vertices belong to another one. Therefore, such an 8–cycle must have four vertices in each of the two copies.

$(4+4)$–situation. Suppose that four vertices π^{s_1}, π^{s_2}, π^{s_3}, π^{s_4} of such an 8–cycle belong to a copy $P_{k-1}(s)$, and other four vertices π^{p_1}, π^{p_2}, π^{p_3}, π^{p_4} belong to a copy $P_{k-1}(p)$. Herewith, vertices π^{s_l}, $l = 1, 2, 3, 4$, should form a path of length three whose endpoints π^{s_1} and π^{s_4} should be adjacent to vertices from a copy $P_{k-1}(p)$. This means that the equality $\pi_1^{s_1} = \pi_1^{s_4} = p$ should hold, and both vertices should belong to the efficient dominating set D_p. So, one vertex of $P_{k-1}(s)$, that is adjacent to a vertex of $P_{k-1}(p)$, must have the form $[p\,\alpha\,q\,\beta\,s]$.

Let $\pi^{s_1} = [p\,\alpha\,q\,\beta\,s]$, where $\pi_1^{s_1} = p$, $\pi_j^{s_1} = q$, $|\alpha| = j-2$, $|\beta| = k-j-1$. By Lemma 2.7.6, a path of length three between vertices is presented by the following two ways.

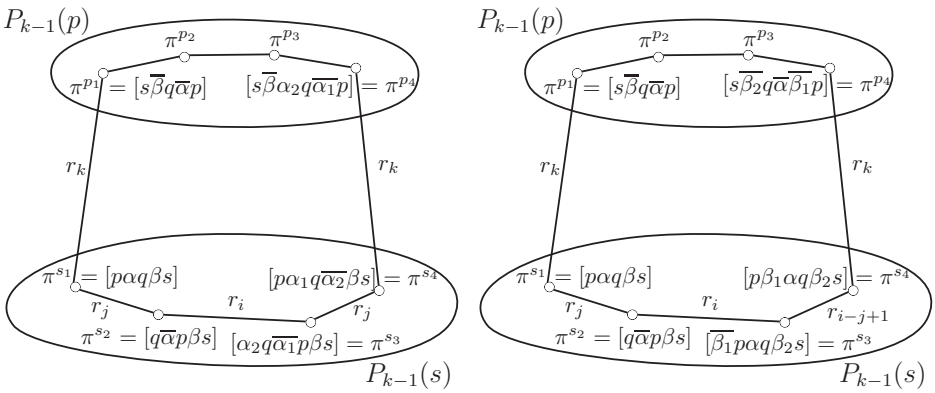


Figure 14. $(4+4)$–situation

1) In the first way, a path of length three has the following form:

$$\pi^{s_1} = [p\alpha q\beta s] \xrightarrow{r_j} \pi^{s_2} = [q\overline{\alpha}p\beta s] \xrightarrow{r_i} \pi^{s_3} = [\alpha_2 q\overline{\alpha_1}p\beta s] \xrightarrow{r_j} \pi^{s_4} = [p\alpha_1 q\overline{\alpha_2}\beta s],$$

where $\pi^{s_2}_j = p$, $\pi^{s_3}_i = q$, $\alpha = \alpha_1\alpha_2$ and $|\alpha_2| = i-1 \geqslant 1$, $\pi^{s_4}_{j-i+1} = q$. The endpoints π^{s_1}, π^{s_4} are adjacent to the following vertices:

$$\pi^{p_1} = \pi^{s_1} r_k = [s\overline{\beta}q\overline{\alpha_2\alpha_1}p]$$

with $\pi^{p_1}_{k-j+1} = q$, and

$$\pi^{p_4} = \pi^{s_4} r_k = [s\overline{\beta}\alpha_2 q\overline{\alpha_1}p]$$

with $\pi^{p_4}_{k-i+j} = q$ (see Figure 14, Case 1).

Let us describe a path of length three between vertices π^{p_1} and π^{p_4} belonging to the same copy $P_{k-1}(p)$. Denote $\gamma_1 = s\overline{\beta}$, $A = q\overline{\alpha_2}$, $B = \alpha_1$, $\gamma_2 = p$, where $|\gamma_1| = |\beta| + 1 \geqslant 1$, $|A| \geqslant 2$, $|B| \geqslant 0$, then permutations π^{p_1}, π^{p_4} take the forms $[\gamma_1 AB\gamma_2]$ and $[\gamma_1\overline{A}B\gamma_2]$. By Lemma 2.7.7 (case 1a), there is a unique path of length three between these permutations if $|\gamma_1| = |\beta| + 1 = k - j \geqslant 1$ and $|A| = |\alpha_2| + 1 = i \geqslant 3$, or $k - j \geqslant 2$ and $i \geqslant 2$, and by Lemma 2.7.7 (case 1b), there are two distinct paths if $k - j = 1$ and $i = 2$.

Hence, such an 8–cycle takes the form $C^1_8 = r_{k-j+i}r_i r_{k-j+i} r_k r_j r_i r_j r_k$, where $2 \leqslant i < j \leqslant k-1$, the canonical form of which corresponds to (2.4) in

Theorem 2.7.4 for $4 \leqslant k \leqslant n$. The case of $k-j=1$ and $i=2$ by symmetry gives one additional form $C_8^2 = r_{k-1}r_2r_{k-1}r_kr_2r_3r_2r_k$, the canonical form of which corresponds to (2.5) in Theorem 2.7.4 for $4 \leqslant k \leqslant n$.

2) In the second way, a path of length three has the following form:

$$\pi^{s_1} = [p\alpha q\beta s] \stackrel{r_j}{\to} \pi^{s_2} = [q\overline{\alpha}p\beta s] \stackrel{r_i}{\to} \pi^{s_3} = [\overline{\beta_1}p\alpha q\beta_2 s] \stackrel{r_{i-j+1}}{\to} \pi^{s_4} = [p\beta_1\alpha q\beta_2 s],$$

where $\pi_j^{s_2} = p$, $\pi_i^{s_3} = q$, $\beta = \beta_1\beta_2$, $|\beta_1| = i-j \geqslant 1$, $|\beta_2| = k-i-1$, $\pi_i^{s_4} = q$. The endpoints π^{s_1}, π^{s_4} are adjacent to the following vertices:

$$\pi^{p_1} = \pi^{s_1}r_k = [s\overline{\beta_2\beta_1}q\overline{\alpha}p],$$

with $\pi_{k-j+1}^{p_1} = q$, and

$$\pi^{p_4} = \pi^{s_4}r_k = [s\overline{\beta_2}q\overline{\alpha}\overline{\beta_1}p],$$

with $\pi_{k-i+1}^{p_4} = q$ (see Figure 14, Case 2).

Let us describe paths of length three between π^{p_1} and π^{p_4} belonging to the same copy $P_{k-1}(p)$. Denote $\gamma_1 = s\overline{\beta_2}$, $A = \overline{\beta_1}$, $B = q\overline{\alpha}$, $\gamma_2 = p$, where $|\gamma_1| = |\beta_2| + 1 \geqslant 1$, $|A| = |\beta_1| \geqslant 1$, $|B| = |\alpha| + 1 \geqslant 1$, then $\pi^{p_1} = [\gamma_1 AB\gamma_2]$, $\pi^{p_4} = [\gamma_1 BA\gamma_2]$. By Lemma 2.7.7 (case 2a), there is a unique path of length three between π^{p_1} and π^{p_4} if $|\gamma_1| = k-i \geqslant 2$, $|A| = |\beta_1| = i-j = 1$ and $|B| = |\alpha| + 1 = j - 1 = 1$. Hence, $j = 2$, $i = 3$, and for $k \geqslant 5$ an 8–cycle has the form $r_{k-1}r_2r_{k-1}r_kr_2r_3r_2r_k$, corresponding again to the canonical form (2.5).

By Lemma 2.7.7 (case 2b), there also exists a unique path of length three between π^{p_1} and π^{p_4} in the case when $|\gamma_1| = k-i = 1$, $|A| = |\beta_1| = i-j \geqslant 1$, $|B| = |\alpha| + 1 = j - 1 \geqslant 1$. This means that $i = k-1$, and such an 8–cycle has the form $C_8^4 = r_kr_{k-j}r_{k-1}r_jr_kr_{k-j}r_{k-1}r_j$, $2 \leqslant j \leqslant k-2$, the canonical form of which corresponds to (2.6) in Theorem 2.7.4, if we set $j = i$. So, there is a unique path of length three under the conditions listed, unless $|A| = |B| = 1$ when by Lemma 2.7.7 (case 2c) this path is not unique. So, $k = 4, j = 2, i = 3$ and 8–cycles take forms $r_2r_3r_2r_4r_3r_2r_3r_4$ and $r_3r_2r_3r_4r_3r_2r_3r_4$, corresponding to forms (2.5) and (2.4).

Thus, the case of two copies is considered and all 8–cycles occurring in this case are found.

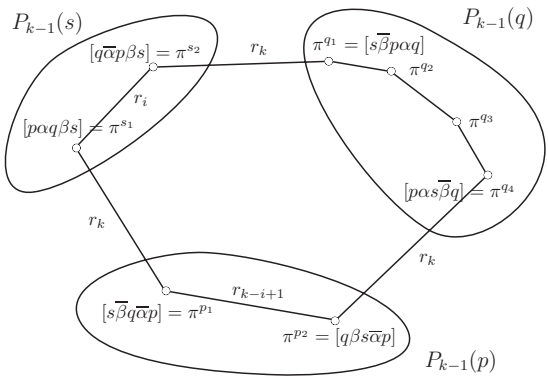


Figure 15. $(2+2+4)$–situation

Case 2: an 8**–cycle within** P_k **has vertices from three copies of** P_{k-1}

Suppose an 8–cycle is formed on vertices from copies $P_{k-1}(p)$, $P_{k-1}(q)$, $P_{k-1}(s)$, where $1 \leqslant p \neq q \neq s \leqslant k$. We have the following possible situations in this case.

$(2+2+4)$–situation. Suppose two vertices π^{s_1}, π^{s_2} of such an 8–cycle belong to a copy $P_{k-1}(s)$, two vertices π^{p_1}, π^{p_2} belong to a copy $P_{k-1}(p)$, and four vertices $\pi^{q_l}, l = 1, \ldots, 4$, belong to a copy $P_{k-1}(q)$ (see Figure 15). Let us consider $\pi^{s_1} = [p\,\alpha\,q\,\beta\,s]$, where $\alpha = [\pi_2 \ldots \pi_{i-1}]$, $|\alpha| = i-2$, and $\beta = [\pi_{i+1} \ldots \pi_{k-1}]$, $|\beta| = k-i-1$. Since a vertex π^{s_2} should be adjacent to a vertex from a copy $P_{k-1}(q)$, hence $\pi_1^{s_2} = q$ and we immediately have

$$\pi^{s_2} = \pi^{s_1} r_i = [q\,\overline{\alpha}\,p\,\beta\,s], \text{ where } \pi_i^{s_2} = p.$$

The vertex π^{s_2} must be joined by an external edge with a vertex π^{q_1} from a copy $P_{k-1}(q)$, hence

$$\pi^{q_1} = \pi^{s_2} r_k = [s\,\overline{\beta}\,p\,\alpha\,q], \text{ where } \pi_{k-i+1}^{q_1} = p.$$

The vertex π^{s_1} must be joined by an external edge with a vertex π^{p_1} from a copy $P_{k-1}(p)$, hence

$$\pi^{p_1} = \pi^{s_1} r_k = [s\,\overline{\beta}\,q\,\overline{\alpha}\,p], \text{ where } \pi_{k-i+1}^{q_1} = s.$$

Since a vertex π^{p_2} should be adjacent to a vertex from a copy $P_{k-1}(q)$, then $\pi_1^{p_2} = q$ and we immediately have

$$\pi^{p_2} = \pi^{p_1} r_{k-i+1} = [q\,\beta\,s\,\overline{\alpha}\,p], \text{ where } \pi_{k-i+1}^{p_2} = s.$$

The vertex π^{p_2} must be joined by an external edge with a vertex π^{q_4} from a copy $P_{k-1}(q)$, hence

$$\pi^{q_4} = \pi^{p_2} r_k = [p\,\alpha\,s\,\overline{\beta}\,q], \text{ where } \pi_i^{q_4} = s.$$

Now we describe a path of length three between vertices π^{q_1} and π^{q_4} which are differed in the order of segments $s\,\overline{\beta}$ and $p\,\alpha$. This means they have the forms $[\gamma_1\,A\,B\,\gamma_2]$ and $[\gamma_1\,B\,A\,\gamma_2]$, where γ_1 is empty, i.e. $|\gamma_1| = 0$, and $A = s\,\overline{\beta}$, $B = p\,\alpha$, $\gamma_2 = q$ and $|A| = |\beta| + 1 \geqslant 1$, $|B| = |\alpha| + 1 \geqslant 1$. Then by Lemma 2.7.7 (case 2d), between vertices π^{q_1} and π^{q_4} there exists a unique path of length three provided that $|A| = k - i \geqslant 2$ and $|B| = i - 1 \geqslant 2$, and no path of this length if $|A| = 1$ or $|B| = 1$. This means that $3 \leqslant j \leqslant k-2$, where $k \geqslant 5$. Thus, an 8–cycle has the following form:

$$C_8^4 = r_k\,r_{k-i+1}\,r_k\,r_{i-1}\,r_{k-1}\,r_{k-i}\,r_k\,r_i, \text{ where } 3 \leqslant i \leqslant k-2,$$

the canonical form of which corresponds to (2.7) in Theorem 2.7.4 for $4 \leqslant k \leqslant n$.

$\underline{(2+3+3)\text{–situation}}$. Suppose two vertices π^{s_1}, π^{s_2} of such an 8–cycle belong to a copy $P_{k-1}(s)$, three vertices $\pi^{p_1}, \pi^{p_2}, \pi^{p_3}$ belong to a copy $P_{k-1}(p)$, and three vertices $\pi^{q_1}, \pi^{q_2}, \pi^{q_3}$ belong to a copy $P_{k-1}(q)$ (see Figure 16). Let $\pi^{s_1} = [p\alpha q\beta s]$, where $\alpha = [\pi_2 \ldots \pi_{i-1}]$, $|\alpha| = i-2$, and $\beta = [\pi_{i+1} \ldots \pi_{k-1}]$, $|\beta| = k-i-1$. Vertices π^{s_1} and π^{s_2} should be adjacent to vertices from distinct $(k-1)$–copies, hence they are presented as follows:

$$\pi^{s_2} = \pi^{s_1} r_i = [q\,\overline{\alpha}\,p\,\beta\,k], \text{ where } \pi_i^{s_2} = p.$$

The vertex π^{s_1} must be joined by an external edge with a vertex π^{p_1} from a copy $P_{k-1}(p)$, hence

$$\pi^{p_1} = \pi^{s_1} r_k = [s\,\overline{\beta}\,q\,\overline{\alpha}\,p], \text{ where } \pi_{k-i+1}^{p_1} = q.$$

The vertex π^{s_2} must be joined by an external edge with a vertex π^{q_1} from a copy $P_{k-1}(q)$, hence

$$\pi^{q_1} = \pi^{s_2} r_k = [s\,\overline{\beta}\,p\,\alpha\,q], \text{ where } \pi_{k-i+1}^{q_1} = p.$$

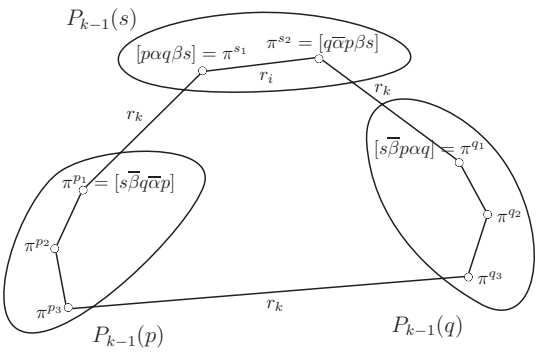


Figure 16. $(2+3+3)$–situation

The vertex π^{p_3} must be joined by an external edge with a vertex π^{q_3} from a copy $P_{k-1}(q)$, hence we have $\pi_1^{p_3} = q$. Thus, vertices π^{p_1} and π^{p_3} should be joined by a path of length two with the condition above. It can be obtained by the following two ways:

$$\pi^{p_1} = [s\overline{\beta}q\overline{\alpha}p] \to \begin{cases} [\beta_2 s\overline{\beta_1}q\overline{\alpha}p] \to [q\beta_1 s\overline{\beta_2}\overline{\alpha}p] = \pi^{p_3^1}, & \text{where } |\beta_2| \neq 0. \\ [\alpha_2 q\beta s\overline{\alpha_1}p] \to [q\overline{\alpha_2}\beta s\overline{\alpha_1}p] = \pi^{p_3^2}, & \text{where } |\alpha_2| \neq 0. \end{cases}$$

From the other side, the vertex π^{q_3} must be joined by an external edge with a vertex $\pi^{p_3} \in P_{k-1}(p)$, hence we have $\pi_1^{q_3} = p$. Thus, vertices π^{q_1} and π^{q_3} should be joined by a path of length two with the condition above. It can be obtained by the following two ways:

$$\pi^{q_1} = [s\overline{\beta}p\alpha q] \to \begin{cases} [\beta_2 s\overline{\beta_1}p\alpha q] \to [p\beta_1 s\overline{\beta_2}\alpha q] = \pi^{q_3^1}, & \text{where } |\beta_2| \neq 0. \\ [\overline{\alpha_1}p\beta s\alpha_2 q] \to [p\alpha_1\beta s\alpha_2 q] = \pi^{q_3^2}, & \text{where } |\alpha_1| \neq 0. \end{cases}$$

Let us note that vertices π^{p_3} and π^{q_3} should be joined by an external edge, which means that the order of segments in the permutations should be reversed. An analysis of non–empty segments in these permutations shows that external edges occur between the following vertices:

- $\pi^{p_3^1}$ and $\pi^{q_3^2}$, when $|\alpha_2| = 0$ and $|\beta_1| = 0$;
- $\pi^{p_3^2}$ and $\pi^{q_3^1}$, when $|\alpha_1| = 0$ and $|\beta_1| = 0$;
- $\pi^{p_3^2}$ and $\pi^{q_3^2}$, when $|\beta| = 0$.

There is no external edge between permutations $\pi^{p_3^1}$ and $\pi^{q_3^1}$ since they have the same order of elements in the segment $s\,\overline{\beta_2}$.

Thus, since $|\alpha| = i-2$, $|\beta| = k-i-1$, then by using the external edge between $\pi^{p_3^1}$ and $\pi^{q_3^2}$, where $|\alpha_2| = 0$ and $|\beta_1| = 0$, we have $|\alpha| \geqslant 1$, $|\beta| \geqslant 1$, and such an 8–cycle has the following form:

$$C_8^5 = r_k\, r_{k-i}\, r_{k-i+1}\, r_k\, r_{i-1}\, r_{k-1}\, r_k\, r_i, \text{ where } 3 \leqslant i \leqslant k-2,\ 5 \leqslant k \leqslant n,$$

the canonical form of which corresponds to (2.8) in Theorem 2.7.4.

Moreover, using the external edge between $\pi^{p_3^2}$ and $\pi^{q_3^1}$, where $|\alpha_1| = 0$ and $|\beta_1| = 0$, we have $|\alpha| \geqslant 1$, $|\beta| \geqslant 1$, and such an 8–cycle has the form $r_k\, r_{k-1}\, r_{i-1}\, r_k\, r_{k-i+1}\, r_{k-i}\, r_k\, r_i$, where $3 \leqslant i \leqslant k-2$, $k \geqslant 5$, the canonical form of which also corresponds to (2.8) in Theorem 2.7.4. Finally, using the external edge between $\pi^{p_3^2}$ and $\pi^{q_3^2}$, where $|\beta| = 0$, we have $i = k-1$, $|\alpha_1| = j \geqslant 1$ and $|\alpha_2| = k-3-j \geqslant 1$, so we have one more 8–cycle of the following form:

$$C_8^6 = r_k\, r_{k-j-1}\, r_{k-j-2}\, r_k\, r_{j+1}\, r_{j+2}\, r_k\, r_{k-1}, \text{ where } 1 \leqslant j \leqslant k-4,\ k \geqslant 5,$$

the canonical form of which corresponds to (2.9) in Theorem 2.7.4 if we put $j = i-1$.

Thus, the case of three copies is considered and all 8–cycles occurring in this case are found.

Case 3: an 8–cycle within P_k has vertices from four copies of P_{k-1}

Suppose that such an 8–cycle occurs on vertices π^{s_1}, π^{s_2} of a copy $P_{k-1}(s)$, π^{t_1}, π^{t_2} of a copy $P_{k-1}(t)$, π^{p_1}, π^{p_2} of a copy $P_{k-1}(p)$, π^{q_1}, π^{q_2} of a copy $P_{k-1}(q)$, where $1 \leqslant s \neq t \neq p \neq q \leqslant k$ (see Figure 17).

Let $\pi^{q_1} = [s\,\alpha\,t\,\beta\,p\,\gamma\,q]$, where $|\alpha| \geqslant 0$, $|\beta| \geqslant 0$, $|\gamma| \geqslant 0$. There are two ways of distributing vertices among four $(k-1)$–copies.

1) Suppose that the vertex π^{q_1} is adjacent to a vertex π^{s_1}, and a vertex π^{q_2} is adjacent to a vertex π^{t_1}. Then vertices π^{s_1} and π^{q_2} should be presented as follows:

$$\pi^{s_1} = \pi^{q_1}\, r_k = [q\,\overline{\gamma}\,p\,\overline{\beta}\,t\,\overline{\alpha}\,s], \qquad \pi^{q_2} = \pi^{q_1}\, r_{|\alpha|+2} = [t\,\overline{\alpha}\,s\,\beta\,p\,\gamma\,q],$$

and hence a vertex π^{t_1}, which is adjacent to π^{q_2}, has the following form:

$$\pi^{t_1} = \pi^{q_2}\, r_k = [q\,\overline{\gamma}\,p\,\overline{\beta}\,s\,\alpha\,t].$$

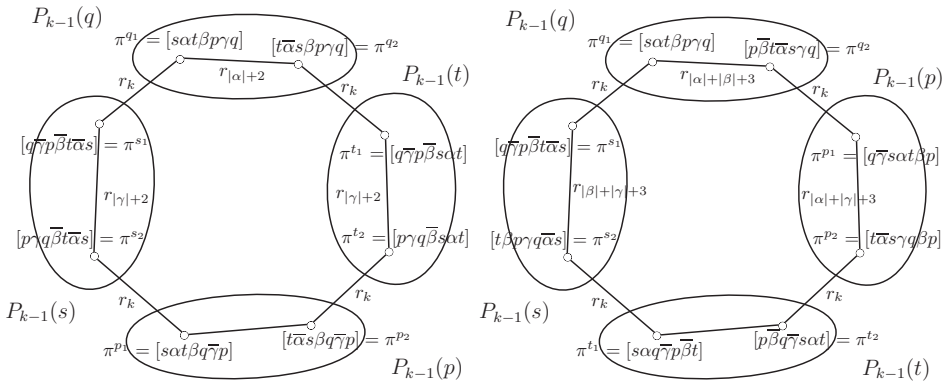


Figure 17. $(2+2+2+2)$–situation

A vertex π^{s_2} should be joined by an external edge with a vertex π^{p_1} of a copy $P_{k-1}(p)$. This means that $\pi_1^{s_2} = p$ and we have:

$$\pi^{s_2} = \pi^{s_1}\, r_{|\gamma|+2} = [p\,\gamma\,q\,\overline{\beta}\,t\,\overline{\alpha}\,s],$$

and a vertex π^{t_2} should be joined by an external edge with a vertex π^{p_2} of a copy $P_{k-1}(p)$. This means that $\pi_1^{t_2} = p$ and we have:

$$\pi^{t_2} = \pi^{t_1}\, r_{|\gamma|+2} = [p\,\gamma\,q\,\overline{\beta}\,s\,\alpha\,t].$$

Vertices π^{s_2} and π^{t_2} should be joined by external edges with vertices π^{p_1} and π^{p_2}, correspondingly, such that:

$$\pi^{p_1} = \pi^{s_2}\, r_k = [s\,\alpha\,t\,\beta\,q\,\overline{\gamma}\,p], \qquad \pi^{p_2} = \pi^{t_2}\, r_k = [t\,\overline{\alpha}\,s\,\beta\,q\,\overline{\gamma}\,p].$$

If an 8–cycle does exist, then vertices π^{p_1}, π^{p_2} should be incident to the same internal edge, and hence, there should exist a prefix–reversal transforming π^{p_1} into π^{p_2}. It is obvious that such a prefix–reversal exists, namely, $r_{|\alpha|+2}$. If we set $|\alpha| = i-2$, $|\beta| = j-i-1$, $|\gamma| = k-j-1$, where $2 \leqslant i < j < k$, then such an 8–cycle has the following form:

$$C_8^7 = r_k\, r_{k-j+1}\, r_k\, r_i\, r_k\, r_{k-j+1}\, r_k\, r_i, \text{ where } 2 \leqslant i < j \leqslant k-1,\ 4 \leqslant k \leqslant n,$$

the canonical form of which corresponds to (2.10) in Theorem 2.7.4.

2) Suppose that the vertex π^{q_1} is adjacent to a vertex π^{s_1}, and a vertex π^{q_2} is adjacent to a vertex π^{s_1}. Then vertices π^{s_1} and π^{q_2} should be presented as follows:

$$\pi^{s_1} = \pi^{q_1} r_k = [q\,\overline{\gamma}\,p\,\overline{\beta}\,t\,\overline{\alpha}\,s], \qquad \pi^{q_2} = \pi^{q_1} r_{|\alpha|+|\beta|+3} = [p\,\overline{\beta}\,t\,\overline{\alpha}\,s\,\gamma\,q].$$

Hence the vertex π^{p_1}, which is adjacent to π^{q_2}, has the following form:

$$\pi^{p_1} = \pi^{q_2} r_k = [q\,\overline{\gamma}\,s\,\alpha\,t\,\beta\,p].$$

A vertex π^{s_2} should be joined by an external edge with a vertex π^{t_1} from a copy $P_{k-1}(t)$, which means that $\pi_1^{s_2} = t$ and we have:

$$\pi^{s_2} = \pi^{s_1} r_{|\beta|+|\gamma|+3} = [t\,\beta\,p\,\gamma\,q\,\overline{\alpha}\,s].$$

Analogically, a vertex π^{p_2} should be joined by an external edge with a vertex π^{t_2} from a copy $P_{k-1}(t)$, which means that $\pi_1^{p_2} = t$, hence we have:

$$\pi^{p_2} = \pi^{p_1} r_{|\alpha|+|\gamma|+3} = [t\,\overline{\alpha}\,s\,\gamma\,q\,\beta\,p].$$

Vertices π^{s_2} and π^{p_2} should be joined by external edges with vertices π^{t_1} and π^{t_2}, correspondingly, such that:

$$\pi^{t_1} = \pi^{t_2} r_k = [s\,\alpha\,q\,\overline{\gamma}\,p\,\overline{\beta}\,t], \qquad \pi^{t_2} = \pi^{p_2} r_k = [p\,\overline{\beta}\,q\,\overline{\gamma}\,s\,\alpha\,t].$$

In this case, an internal edge between vertices π^{t_1} and π^{t_2} does exist only if $|\alpha| = |\beta| = |\gamma| = 0$, which means that $n = 4$ and such an 8–cycle takes the form:

$$C_8^8 = r_4\,r_3\,r_4\,r_3\,r_4\,r_3\,r_4\,r_3,$$

presented as the form (2.11) in Theorem 2.7.4.

Thus, all canonical forms for 8–cycles in P_n, $n \geqslant 4$, are obtained.

Now we count the total number N_8 of distinct 8–cycles passing through a given vertex in the graph. We denote

$$N_8 = \sum_{i=1}^{8} N_{C_8^i},$$

where $N_{C_8^i}$ corresponds to the number of distinct 8–cycles described by the canonical form C_8^i, $1 \leqslant i \leqslant 8$, and passing through a given vertex. Let us note that any canonical form of an l–cycle describes l cycles (not necessarily

distinct) for a given vertex. Among all canonical forms (2.4)–(2.11), there is the only one, namely the form (2.8), which describes eight distinct 8–cycles. In other cases, identical forms occur. For example, from the canonical form $C_8^8 = r_4r_3r_4r_3r_4r_3r_4r_3$ one can get two forms, namely, $r_4r_3r_4r_3r_4r_3r_4r_3$ and $r_3r_4r_3r_4r_3r_4r_3r_4$ which are identical because they describe the same 8–cycle. Thus, the canonical form C_8^8 gives the only 8–cycle, hence, $N_{C_8^8} = 1$.

Let us consider all other cases and find N_8.

The canonical form $C_8^1 = r_k r_j r_i r_j r_k r_{k-j+i} r_i r_{k-j+i}$, where $2 \leqslant i < j \leqslant k-1$ and $4 \leqslant k \leqslant n$, gives identical forms of 8–cycles when $j = k - j + i$ such that: for a fixed k there are two identical forms for all $j = k - j + i$; for fixed k and i there are only two distinct forms for all $j = k - j + i$. So, the number of distinct 8–cycles is one quarter of the total number forms presented by the canonical form C_8^1:

$$N_{C_8^1} = \frac{8}{4} \sum_{k=1}^{n-3} \sum_{s=1}^{k} s = 2\,\frac{(n-3)\,(n-2)\,(n-1)}{6} = \frac{(n-3)\,(n-2)\,(n-1)}{3}.$$

For any form obtained from the canonical form $C_8^2 = r_k r_{k-1} r_2 r_{k-1} r_k r_2 r_3 r_2$, where $4 \leqslant k \leqslant n$, there exists its reversed form, which means that the number of distinct 8–cycles is half of the total number forms presented by the canonical form C_8^2:

$$N_{C_8^2} = \frac{8(n-3)}{4} = 4(n-3).$$

The canonical form $C_8^3 = r_k r_{k-i} r_{k-1} r_i r_k r_{k-i} r_{k-1} r_i$, where $2 \leqslant i \leqslant k-2$ and $4 \leqslant k \leqslant n$, consists of two identical parts, which means that it gives at least four distinct forms. Moreover, if k is even then for all $i = \frac{k}{2}$ there are only two distinct forms. We count numbers m_1 and m_2 of distinct forms obtained from the canonical form C_8^3 for all even $2s + 2 \leqslant n$ and all odd $2s + 3 \leqslant n$, respectively. Using identical forms we have:

$$m_1 = 4 \sum_{s=1}^{\frac{n-2}{2}} s - 2\left(\frac{n}{2} - 1\right) = \frac{n(n-2)}{2} - (n-2) = \frac{(n-2)^2}{2},$$

$$m_2 = 4 \sum_{s=1}^{\frac{n-3}{2}} s = \frac{(n-3)(n-1)}{2}.$$

If n is even then $N_{C_8^3} = \frac{(n-2)^2}{2} + \frac{((n-1)-3)((n-1)-1)}{2} = (n-2)(n-3)$, and if n is odd then $N_{C_8^3} = \frac{(n-3)(n-1)}{2} + \frac{((n-1)-2)^2}{2} = (n-2)(n-3)$, hence for any $n \geqslant 4$ we have

$$N_{C_8^3} = (n-2)(n-3).$$

Similar counts we use for the form $C_8^4 = r_k r_{k-i+1} r_k r_i r_k r_{k-i} r_{k-1} r_{i-1}$, where $2 \leqslant i \leqslant k-2$ and $5 \leqslant k \leqslant n$, which gives identical forms of 8–cycles when $i = k-i+1$, such that if k is odd then for all $i = \frac{k+1}{2}$ there are only four distinct forms. Using identical forms we count the numbers m_1 and m_2 of distinct forms obtained from the canonical form C_8^4 for all even $2s+4 \leqslant n$ and all odd $2s+3 \leqslant n$, correspondingly:

$$m_1 = 8 \sum_{s=1}^{\frac{n-4}{2}} s = (n-2)(n-4),$$

$$m_2 = 8 \sum_{s=1}^{\frac{n-3}{2}} s - 4\left(\frac{n-1}{2} - 1\right) = (n-1)(n-3) - 2(n-3) = (n-3)^2.$$

If n is even then $N_{C_8^4} = (n-2)(n-4) + ((n-1)-3)^2 = 2(n-3)(n-4)$, and if n is odd then $N_{C_8^4} = (n-3)^2 + ((n-1)-2)((n-1)-4) = 2(n-3)(n-4)$, hence for any $n \geqslant 5$ we have

$$N_{C_8^4} = 2(n-3)(n-4).$$

The canonical form $C_8^5 = r_k r_{k-1} r_{i-1} r_k r_{k-i+1} r_{k-i} r_k r_i$, where $3 \leqslant i \leqslant k-2$, $5 \leqslant k \leqslant n$, gives only distinct forms of 8–cycles, so we have:

$$N_{C_8^5} = 8 \sum_{s=1}^{k-4} s = 4\,(n-3)(n-4).$$

The calculations for the canonical form $C_8^6 = r_k r_{k-1} r_k r_{k-i} r_{k-i-1} r_k r_i r_{i+1}$, where $2 \leqslant i \leqslant k-3$ and $5 \leqslant k \leqslant n$, are similar to those we have for the canonical form C_8^4, so for any $n \geqslant 5$ we have:

$$N_{C_8^6} = 2(n-3)(n-4).$$

The canonical form $C_8^7 = r_k r_{k-j+1} r_k r_i r_k r_{k-j+1} r_k r_i$, where $2 \leqslant i < j \leqslant k-1$ and $4 \leqslant k \leqslant n$, consists of two identical parts, so it gives at most four

	sequence of vertices	algebraic description
1	1234-4321-2341-3241-4231-1324-3124-2134	$r_4\, r_3\, r_2\, r_3\, r_4\, r_2\, r_3\, r_2$
2	1234-3214-2314-1324-4231-2431-3421-4321	$r_3\, r_2\, r_3\, r_4\, r_2\, r_3\, r_2\, r_4$
3	1234-2134-3124-4213-2413-1423-4123-3214	$r_2\, r_3\, r_4\, r_2\, r_3\, r_2\, r_4\, r_3$
4	1234-2134-4312-1342-3142-4132-2314-3214	$r_2\, r_4\, r_3\, r_2\, r_3\, r_4\, r_2\, r_3$
5	1234-4321-2341-3241-4231-1324-2314-3214	$r_4\, r_3\, r_2\, r_3\, r_4\, r_3\, r_2\, r_3$
6	1234-2134-3124-4213-1243-2143-4123-3214	$r_2\, r_3\, r_4\, r_3\, r_2\, r_3\, r_4\, r_3$
7	1234-4321-3421-2431-4231-1324-3124-2134	$r_4\, r_2\, r_3\, r_2\, r_4\, r_2\, r_3\, r_2$
8	1234-3214-2314-4132-1432-3412-4312-2134	$r_3\, r_2\, r_4\, r_2\, r_3\, r_2\, r_4\, r_2$
9	1234-4321-3421-1243-2143-3412-4312-2134	$r_4\, r_2\, r_4\, r_2\, r_4\, r_2\, r_4\, r_2$
10	1234-4321-2341-1432-3412-2143-4123-3214	$r_4\, r_3\, r_4\, r_3\, r_4\, r_3\, r_4\, r_3$

Table 2. 8–cycles in P_4 containing the identity permutation $I = [1234]$.

identical forms. However, if $i = k - j + 1$ then all forms are identical. So, the number of distinct 8–cycles is one–eighth of the total number of forms presented by the canonical form C_8^7 and we have:

$$N_{C_8^7} = \frac{8}{8} \sum_{k=1}^{n-3} \sum_{s=1}^{k} s = \frac{(n-3)\,(n-2)\,(n-1)}{6}.$$

Thus, summing all numbers for $N_{C_8^i}$, $1 \leqslant i \leqslant 8$, we get the total number of distinct 8–cycles passing through a given vertex in the graph:

$$N_8 = \frac{n^3 + 12n^2 - 103n + 176}{2}.$$

This completes the proof of the main statement of Theorem.

The data produced by the given formula for N_8 coincides with the data produced by a computer experiment. For example, there are 10, 43 and 103 distinct 8–cycles passing through each vertex for $n = 4$, 5 and 6. These data were obtained by M. Orlov, Novosibirsk State University. All 8–cycles passing through $I = [1234]$ in P_4 are presented in Table 2.

The total number of distinct 8–cycles in P_n, $n \geqslant 4$, is obtained as follows. There are $n!$ vertices in the graph, each of which belongs to N_8 distinct 8–cycles. Hence, in total there are $\frac{n!\,N_8}{8} = \frac{n!(n^3+12n^2-103n+176)}{16}$ distinct 8–cycles in the graph. In particular, there are 30 cycles of length eight in P_4.

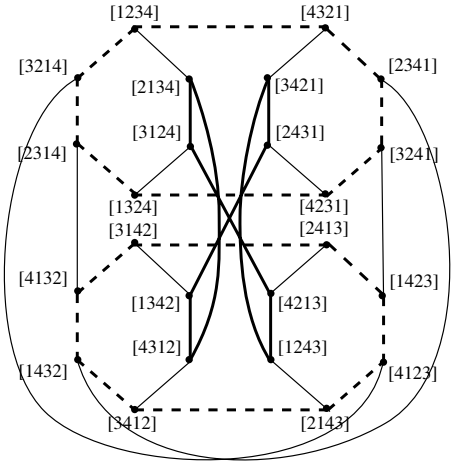


Figure 18. Independent 8–cycles in P_4

To show that there are $\frac{n!}{8}$ independent 8–cycles in P_n, $n \geqslant 4$, we use the hierarchical structure of the Pancake graph. In P_4 there are three independent 8–cycles presented, for instance, as:

$$C_8^1 = [1234] - [4321] - [2341] - [3241] - [4231] - [1324] - [2314] - [3214],$$

$$C_8^2 = [2134] - [4312] - [1342] - [2431] - [3421] - [1243] - [4213] - [3124],$$

$$C_8^3 = [3124] - [2413] - [1423] - [4123] - [2143] - [3412] - [1432] - [4132].$$

These independent 8–cycles are shown in Figure 18. From the hierarchical structure of P_n, it follows that there are $\frac{n!}{24}$ copies of P_4, each of which consists of exactly three independent 8–cycles. Hence, in total we have $\frac{n!}{8}$ independent 8–cycles in P_n. This completes the proof of Theorem. □

As one can see, the Pancake graph has a very complicated cycle structure. This makes one of the main difficulties in solving the Pancake problem that is the problem of finding the diameter of the Pancake graph. This problem is still open. We consider this problem in the Section 3.2.

Chapter 3

The diameter problem

Cayley graphs tend to have a number of other desirable properties as well, including low diameter. There is the problem of establishing the diameter of a Cayley graph $\Gamma = Cay(G, S)$, that is the maximum, over $g \in G$, of the length of a shortest expression for g as a product of generators. Computing the diameter of an arbitrary Cayley graph over a set of generators is NP–hard since the minimal word problem is known to be NP–hard in general. This result was shown in 1981 by Shimon Even and Oded Goldreich in [30]. The diameter problem appears in the *Rubik's cube* puzzle. Recently it was announced [34] that every position of Rubik's Cube can be solved in twenty moves or less which represents the diameter of a corresponding Cayley graph. General upper and lower bounds are very difficult to obtain. Moreover, there is a fundamental difference between Cayley graphs of abelian and non–abelian groups.

3.1 Diameter of Cayley graphs on abelian and non–abelian groups

Laszlo Babai et al. [7] have considered in 1989 the diameter of Cayley graphs on non–abelian finite simple groups and the following result was obtained.

Theorem 3.1.1 [7] *Every non–abelian finite simple group G has a set of $\leqslant 7$ generators such that the resulting Cayley graph has diameter $O(\log_2 |G|)$.*

So, they have shown that each non–abelian simple group has a set of at most seven generators that yields a Cayley graph with logarithmic diameter

(with constant factors). However, this property does not hold for Cayley graphs of abelian groups as it was shown in 1993 by Fred Annexstein and Marc Baumslag in [3]. They have found a lower bound on the diameter of Cayley graphs on abelian groups. Let us recall, that in an abelian group the result of applying the group operation to two group elements does not depend on their order, i.e. for any $g, h \in G$ we have $gh = hg$, and every subgroup of an abelian group is normal.

By $diam(Cay(G, S))$ we denote the diameter of a Cayley graph $Cay(G, S)$ on a group G with a generating set S. The result presented below is actually obtained for a Cayley digraph, which means that there are no restrictions for a generating set to be a symmetric or identity free.

Theorem 3.1.2 [3] *Let G be an abelian group with a generating set S of size r. The Cayley graph $Cay(G,\, S)$ has the following diameter bound:*

$$diam(Cay(G,\, S)) \geqslant \frac{1}{e}\,|G|^{1/r}.$$

Proof. Suppose that G is an abelian group with a generating set S of size r. Let n be the number of group elements that can be written as a product of $\leqslant d = diam(Cay(G,\, S))$ elements from S. Since G is abelian, n is at most equal to the number of ways d objects can be chosen from the set of $(r+1)$ objects with repetition allowed. The set of $(r+1)$ objects is the set S with the identity element. The number of ways n is bounded as follows:

$$n \leqslant \binom{r+d}{d} = \frac{(r+d)!}{r!\,d!} \leqslant \frac{(r+d)^r}{r!} \leqslant \frac{(r\,d)^r}{r!} \leqslant (ed)^r.$$

Solving for d yields the given bound. □

We can get a tighter bound in the case $r \leqslant |G|^{1/r} \leqslant d$. For then:

$$n \leqslant \binom{r+d}{d} = \frac{(r+d)!}{r!\,d!} \leqslant \frac{(r+d)^r}{r!} \leqslant \frac{(2d)^r}{r!} \leqslant \left(\frac{2ed}{r}\right)^r.$$

On the other hand, in 1988 it was conjectured by Laszlo Babai and Akos Seress [9] for non–abelian groups that the diameter will always be small.

Conjecture 3.1.3 [9] *There exist a constant c such that for every non–abelian finite simple group G, the diameter of every Cayley graph of G is $\leqslant (\log_2 |G|)^c$.*

If the conjecture is true, one would expect to find Cayley graphs of these groups with small diameter. But this problem is open even for the *alternating groups* A_n consisting of all even permutations on $\{1, \ldots, n\}$. The first step towards a solution this conjecture was made by Babai and Seress [9] for the symmetric Sym_n and alternating A_n groups.

Theorem 3.1.4 [9] *If G is either Sym_n or A_n then the diameter of every Cayley graph of G is $\leqslant exp((n \ln n)^{(1/2)}(1 + o(1)))$.*

Even for simple examples the exact diameter is unknown and there are only bounds. For example, the diameter of the Pancake graph is unknown.

3.2 Pancake problem

The original (unburnt) Pancake problem was posed in 1975 in the American Mathematical Monthly by Jacob E. Goodman [28] writing under the name "Harry Dweighter" (or "Harried Waiter") and it is stated as follows:

> "The chef in our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest on the bottom) by grabbing several pancakes from the top and flips them over, repeating this (varying the number I flip) as many times as necessary. If there are n pancakes, what is the maximum number of flips (as a function of n) that I will ever have to use to rearrange them?"

It is clear that a stack of these n pancakes can be represented by a permutation on n elements and the problem is to find the minimum number of flips (prefix–reversals) needed to transform a permutation into the identity permutation. Clearly, this number of flips corresponds to the diameter $diam(P_n)$ of the Pancake graph. There is the following open problem.

Problem 3.2.1 *What is the diameter $diam(P_n)$ of the Pancake graph?*

Simple lower and upper bounds for $diam(P_n)$ are obtained as follows. Consider any stack of pancakes. An adjacency in this stack is a pair of pancakes that are adjacent in the stack, and such that no other pancake has size intermediate between two. If the largest pancake is on the bottom, this also counts as one extra adjacency. Now, for $n \geqslant 4$ there are stacks of n pancakes that have no adjacencies and each move (flip) can create at most one adjacency. So, we have $n \leqslant diam(P_n)$. For upper bounds, one can use the following procedure. Given any stack we may start by bringing the largest pancake on top and then flip the whole stack: the largest pancake is now at the bottom, after two flips. Inductively, bring to the top the largest pancake that has not been sorted yet, and then flip it to the bottom of the unsorted stack. So, by $2(n-1)$ flips we will have thus sorted the whole thing, and the upper bound is $diam(P_n) \leqslant 2(n-1)$.

In 1979 W. H. Gates and Ch. H. Papadimitriou [33] presented the improved upper and lower bounds for the diameter of the Pancake graph as $17n/16 \leqslant diam(P_n) \leqslant 5/3(n+1)$. In the next section we give an algorithm for these bounds. Then bounds will be proved due to [33].

3.2.1 An algorithm by Gates and Papadimitrou

Let $\pi = [\pi_1 \dots \pi_n]$ be a permutation and $r_i, 2 \leqslant i \leqslant n$, be a prefix–reversal such that $[\pi_1 \dots \pi_i \, \pi_{i+1} \dots \pi_n] r_i = [\pi_i \dots \pi_1 \, \pi_{i+1} \dots \pi_n]$. If $|\pi_i - \pi_{i+1}| = 1$, $1 \leqslant i \leqslant n-1$, we say that the pair $(i, i+1)$ is an *adjacency* in π. We also consider $(i, i+1)$ to be adjacency if we have a segment $[\pi_i \pi_{i+1}] = [1\, n]$ or $[\pi_{i+1} \pi_i] = [n\, 1]$ in π. Moreover, if $\pi_1 = 1$ and $\pi_n = n$ then the pair $(1, n)$ is also an adjacency. All successive elements of a permutation having adjacency form a *block* in a permutation π. If element π_i is not in a block, i.e. $(i-1, i)$ and $(i, i+1)$ are not adjacencies in π, then it is a *free*. For example, in [654132] there are two blocks [654] and [32], and one free element 1, and in [321654] there are 5 adjacencies, one block (since [16] has an adjacency) and no one free elements.

Let us note that the identity permutation $I_n = [123 \dots n]$ has one block and n adjacencies. So, the presented algorithm sorts the permutation π so as to create a total n adjacencies. Moreover, at each step it is determined by what sequence of prefix–reversals (no more than 4) should be applied to

increase an adjacency in a permutation. In total, there are nine different cases. In the description of the algorithm below we use o to stand for one of $\{-1, 1\}$. Addition is understood modulo n.

The algorithm GP

Step 0. Input: a permutation $\pi = [\pi_1 \pi_2 \dots \pi_n] \neq I_n$.
If a given permutation has $n-1$ adjacencies then go to **Step 2**, otherwise go to **Step 1.**

Step 1. Let $\pi_1 = t$. Then the following cases hold.

Case 1. If t and $\pi_i = t + o$ are free elements then we get a new permutation π^* by the following way:

$$[t \dots \pi_{i-1}\, t + o\, \pi_{i+1} \dots \pi_n]\, r_{i-1} = [\pi_{i-1} \dots t\, t + o\, \pi_{i+1} \dots \pi_n] = \pi^*,$$

i.e. $\pi^* = \pi\, r_{i-1}$.

Case 2. If t is free and $\pi_i = t + o$ is the first element of a block then we get a new permutation π^* as in Case 1: $\pi^* = \pi\, r_{i-1}$.

Case 3. If t is free, $\pi_i = t + o$ and $\pi_j = t - o$ are the last elements of blocks then we get a new permutation π^* by the following way:

$$[t \dots \pi_{i-1}\, t + o \dots \pi_{j-1}\, t - o \dots \pi_n] \overset{r_i}{\to} [t + o\, \pi_{i-1} \dots t \dots \pi_{j-1}\, t - o \dots \pi_n] \overset{r_{i-1}}{\to}$$

$$[\dots t + o\, t \dots \pi_{j-1}\, t - o \dots \pi_n] \overset{r_j}{\to} [t - o\, \pi_{j-1} \dots t\, t + o \dots \pi_n] \overset{r_{j-i}}{\to}$$

$$\to [\dots t - o\, t\, t + o \dots \pi_n] = \pi^*,$$

i.e. a permutation π^* is obtained from a permutation π by applying four prefix–reversals as follows:

$$\pi^* = \pi\, r_i\, r_{i-1}\, r_j\, r_{j-i}.$$

Case 4. If t is in a block and $\pi_i = t + o$ is free then we get a new permutation π^* as in Case 1: $\pi^* = \pi\, r_{i-1}$.

Case 5. If t is in a block and $\pi_i = t + o$ is the first element of a block then we get a new permutation π^* as in Case 1: $\pi^* = \pi\, r_{i-1}$.

Case 6. If t is in a block with the last element $\pi_s = t + k \cdot o$, $k > 0$, $\pi_j = t - o$ is the last element of another block and $\pi_i = t + (k+1) \cdot o$ is free, then depending on the relative position of the two blocks and $t + (k+1) \cdot o$, a new permutation π^* is obtained by either way (a):

$$[t \dots \pi_{s-1}\ t + k \cdot o\ \pi_{s+1} \dots \pi_{i-1}\ t + (k+1) \cdot o\ \pi_{i+1} \dots \pi_{j-1}\ t - o \dots \pi_n] \xrightarrow{r_i}$$
$$[t + (k+1) \cdot o\ \pi_{i-1} \dots \pi_{s+1}\, t + k \cdot o\ \pi_{s-1} \dots t\, \pi_{i+1} \dots \pi_{j-1}\ t - o \dots \pi_n] \xrightarrow{r_{i-s}}$$
$$[\dots \pi_{i-1}\ t + (k+1) \cdot o\ t + k \cdot o\ \pi_{s-1} \dots t\, \pi_{i+1} \dots \pi_{j-1}\ t - o \dots \pi_n] \xrightarrow{r_j}$$
$$[t - o \dots \pi_{i+1}\ t \dots t + k \cdot o\ t + (k+1) \cdot o \dots \pi_n] \xrightarrow{r_{j-i}}$$
$$[\dots t - o\ t \dots t + k \cdot o\ t + (k+1) \cdot o \dots \pi_n] = \pi^*,$$

or way (b):

$$[t \dots \pi_{s-1}\ t + k \cdot o\ \pi_{s+1} \dots \pi_{j-1}\ t - o\ \pi_{j+1} \dots \pi_{i-1}\ t + (k+1) \cdot o\ \pi_{i+1} \dots \pi_n] \xrightarrow{r_i}$$
$$[t + (k+1) \cdot o\ \pi_{i-1} \dots \pi_{j+1}\ t - o\ \pi_{j-1} \dots \pi_{s+1}\ t + k \cdot o\ \pi_{s-1} \dots t \dots \pi_n] \xrightarrow{r_{i-s}}$$
$$[\dots \pi_{j-1}\ t - o\ \pi_{j+1} \dots \pi_{i-1}\ t + (k+1) \cdot o\ t + k \cdot o \dots t\ \pi_{i+1} \dots \pi_n] \xrightarrow{r_i}$$
$$[t \dots t + k \cdot o\ t + (k+1) \cdot o \dots \pi_{j+1}\ t - o \dots \pi_n] \xrightarrow{r_{i-j+s}}$$
$$[\dots t + (k+1) \cdot o\ t + k \cdot o\ \dots t\ t - o \dots \pi_n] = \pi^*.$$

Thus, in Case 6 (a) a permutation π^* is obtained from a permutation π by applying four prefix–reversals as follows:

$$\pi^* = \pi\, r_i\, r_{i-s}\, r_j\, r_{j-i}.$$

In Case 6 (b) a permutation π^* is obtained from a permutation π by also applying four prefix–reversals as follows:

$$\pi^* = \pi r_i r_{i-s} r_i r_{i-j+s}.$$

Case 7. If t is in a block with the last element $\pi_i = t + k \cdot o$, $k > 0$, and $\pi_j = t + (k+1) \cdot o$ is in a block, then depending on whether π_j is at the beginning (a), or the end (b) of its block, a permutation π^* is obtained from a permutation π as follows:

$$(a): \quad [t \dots \pi_{i-1}\ t + k \cdot o\ \pi_{i+1} \dots t + (k+1) \cdot o\ \pi_{j+1} \dots \pi_n] \xrightarrow{r_i}$$
$$[t + k \cdot o\ \pi_{i-1} \dots t\ \pi_{i+1} \dots t + (k+1) \cdot o\ \pi_{j+1} \dots \pi_n] \xrightarrow{r_{j-1}}$$

$$[\dots t \dots t + k \cdot o \; t + (k+1) \cdot o \dots \pi_n] = \pi^*.$$

$$(b): \quad [t \dots \pi_{i-1} \; t + k \cdot o \; \pi_{i+1} \dots \pi_{j-1} \; t + (k+1) \cdot o \dots \pi_n] \stackrel{r_j}{\to}$$

$$[t + (k+1) \cdot o \; \pi_{j-1} \dots \pi_{i+1} \; t + k \cdot o \dots t \dots \pi_n] \stackrel{r_{j-i}}{\to}$$

$$[\dots t + (k+1) \cdot o \; t + k \cdot o \dots t \dots \pi_n] = \pi^*.$$

Thus, in Case 7 (a) a permutation π^* is obtained from a permutation π by using two prefix–reversals:

$$\pi^* = \pi \, r_i \, r_{j-1}.$$

In Case 7 (b) a permutation π^* is also obtained from a permutation π by applying two prefix–reversals as follows:

$$\pi^* = \pi \, r_j \, r_{j-i}.$$

If a permutation π^* has $n-1$ adjacencies then $\pi := \pi^*$ and go to **Step 2**, otherwise $\pi := \pi^*$ and repeat **Step 1.**

Step 2. A permutation π with $n-1$ adjacencies has one block. Then there are two cases.

Case 8. Let $\pi = [i-1 \dots 1 \, n \dots i]$, where $\pi_i = n$, then I_n is obtained by the following way:

$$[i-1 \dots 1 \, n \dots i] \stackrel{r_n}{\to} [i \dots n \, 1 \dots i-1] \stackrel{r_{n-i+1}}{\to} [n \dots i \, 1 \dots i-1] \stackrel{r_n}{\to}$$

$$[i-1 \dots 1 \, i \dots n] \stackrel{r_{i-1}}{\to} [1 \dots i-1 \, i \dots n] = I_n,$$

i.e. the identity permutation is obtained from a permutation π by applying four prefix–reversals such that:

$$I_n = \pi r_n r_{n-i+1} r_n r_{i-1}.$$

Case 9. Let $\pi = [i \dots n \, 1 \dots i-1]$, where $\pi_{n-i+1} = n$, then I_n is obtained by the following way:

$$[i \dots n \, 1 \dots i-1] \stackrel{r_{n-i+1}}{\to} [n \dots i \, 1 \dots i-1] \stackrel{r_n}{\to} [i-1 \dots 1 \, i \dots n] \stackrel{r_{i-1}}{\to}$$

$$[1 \dots i-1 \, i \dots n] = I_n,$$

i.e. the identity permutation is obtained from a permutation π by applying three prefix–reversals such that:

$$I_n = \pi r_{n-i+1} r_n r_{i-1}.$$

The end

3.2.2 Upper bound on the diameter of the Pancake graph

Theorem 3.2.2 [33] *Algorithm GP creates the identity permutation by at most* $(5n+5)/3$ *prefix–reversals.*

Proof. First, it is clear that if we have a permutation with less than $n-1$ adjacencies, one of the Cases 1–7 is applicable. Hence, the algorithm does not halt unless $n-1$ adjacencies have been created. Then one of the Cases 8–9 is applicable to get the identity permutation with n adjacencies and one block. Then the algorithm is finished. Obviously, the algorithm will eventually halt, since at each execution of the main loop at least one new adjacency is created and none are destroyed. It remains to prove that it does so in no more than $(5n+5)/3$ prefix–reversals.

Call the action of Case i as *action of a type* i (or just action i). Let x_i, where $i = 1, \ldots, 9$, denote the number of actions of type i performed by an execution of the algorithm assuming that $x_3 = x_6$. Then the total number z of prefix–reversals given by the Cases 1–7 are presented as follows:

$$z = x_1 + x_2 + 4x_3 + x_4 + x_5 + 2x_7,$$

where x_i is multiplied by the number of prefix–reversals involved in the corresponding case. Action 3 can be divided into four special cases, according to what happens in the corresponding flipping that comes before the last. The top of stack before the flipping and the element next to $t-o$ may either: 1) be non–adjacent; 2) form a new block; 3) merge a block with a singleton; 4) merge two blocks. Accordingly, we distinguish among these subcases by writing $x_3 = x_{31} + x_{32} + x_{33} + x_{34}$.

Now, since each action increases the number of adjacencies as indicated in the table below, the total number of $n-1$ adjacencies in the conclusion of the Step 1 (Cases 1–7) of the algorithm is presented as:

$$n - 1 = a + x_1 + x_2 + 2x_{31} + 3x_{32} + 3x_{33} + 3x_{34} + x_4 + x_5 + x_7, \quad (3.1)$$

where a is a number of adjacencies in a given permutation π.

Finally, if b is the number of blocks in a given permutation π then we have:

$$b + x_1 - x_{31} - x_{33} - 2x_{34} - x_5 - x_7 = 1, \quad (3.2)$$

Action	1	2	31	32	33	34	4	5	7
number of prefix–reversals	1	1	4	4	4	4	1	1	2
increase in adjacencies	1	1	2	3	3	3	1	1	1
increase in number of blocks	1	0	-1	0	-1	-2	0	-1	-1

because each type of actions increases or decreases the number of blocks as indicated in the table above, we start with b blocks and we end up with 1 block. Also notice that $b \leqslant a$, whereby, from (3.1) becomes:

$$x_1 + x_2 + 2x_{31} + 3x_{32} + 3x_{33} + 3x_{34} + x_4 + x_5 + x_7 + b \leqslant n - 1. \quad (3.3)$$

Thus, any possible application of the algorithm would, at worst,

$$\text{maximize} \quad z = x_1 + x_2 + 4x_3 + x_4 + x_5 + 2x_7,$$

subject to (3.2) and (3.3). Then it is *claimed* that the maximum is achieved for the values:

$$x_1 = (n+1)/3,\ x_2 = 0,\ x_3 = x_{31} = (n-2)/3,\ x_4 = x_5 = x_7 = b = 0,$$

yielding a value of z equal to $z = (5n-7)/3$. To show this claim, recall the duality Theorem stating that this maximum value equals the minimum values of the *dual linear program*:

$$\text{minimize} \quad \omega = \xi_2 + (n-1)\xi_3,$$

subject to the inequalities:

$$\begin{aligned}
\xi_2 + \xi_3 &\geqslant 1,\\
\xi_3 &\geqslant 1,\\
-\xi_2 + 2\xi_3 &\geqslant 4,\\
3\xi_3 &\geqslant 4,\\
-\xi_2 + 3\xi_3 &\geqslant 4,\\
-2\xi_2 + 3\xi_3 &\geqslant 4,\\
\xi_3 &\geqslant 1,\\
-\xi_2 + \xi_3 &\geqslant 1,\\
-\xi_2 + \xi_3 &\geqslant 2,\\
\xi_2 + \xi_3 &\geqslant 0,
\end{aligned}$$

where ξ_2 and ξ_3 correspond to increases in number of blocks and in adjacencies, correspondingly (see the table above).

Thus, in order to prove the claim, we just have to exhibit a pair (ξ_2, ξ_3) satisfying these inequalities and having $\omega = \xi_2 + (n-1)\xi_3 = (5n-7)/3$. And such pair is $\xi_2 = -2/3$, $\xi_3 = 5/3$.

Thus, we get a permutation with $n-1$ adjacencies in no more than $(5n-7)/3$ prefix–reversals. The bound $(5n+5)/3$ now follows directly, since it takes four more prefix–reversals to transform a permutation with $n-1$ adjacencies to the identity permutation which follows from the Cases 8–9 of the algorithm. Finally, algorithm GP creates the identity permutation by at most $(5n+5)/3$ prefix–reversals. □

3.2.3 Lower bound on the diameter of the Pancake graph

Let $\tau = [17536428]$. For k, a positive integer, τ_k denotes τ with each of the integers increased by $8(k-1)$. In other words, $\tau_k = [1_k 7_k 5_k 3_k 6_k 4_k 2_k 8_k]$ where $m_k = m + 8(k-1)$. Consider the permutation

$$\chi = \tau_1 \, \tau_2^R \, \tau_3 \, \tau_4^R \, \cdots \, \tau_{m-1} \, \tau_m^R, \tag{3.4}$$

where m is an even integer, $n = |\chi| = 8\,m$, $\tau = \tau_1$, and $\tau_k^R = \tau_k \, r_8 = [8_k 2_k 4_k 6_k 3_k 5_k 7_k 1_k]$ for any even $k \leqslant m$.

Let $f(\chi)$ be the number of prefix–reversals transforming a permutation χ to the identity permutation. Then the following theorem takes place.

Theorem 3.2.3 [33] $17\,n/16 \leqslant f(\chi) \leqslant 19\,n/16$ *for* n *a multiple of* 16.

Proof. To show the upper bound, we first do the following sequences of prefix–reversals:

$$\chi \to \tau_2 \, \tau_1^r \, \tau_3 \, \cdots \to \tau_2^r \, \tau_1^r \, \tau_3 \, \cdots \to \tau_1 \, \tau_2 \, \tau_3 \, \cdots$$

and so on, bringing the even–indexed χ's in front and then back with the reversal canceled in three moves. Thus, in $3n/16$ moves we obtain a permutation $\chi' = \tau_1 \, \tau_2 \, \tau_3 \, \tau_4 \, \cdots \, \tau_{m-1} \, \tau_m$. Then, for each copy of τ in χ' we repeat the following sequence of eight moves (among a number of possibilities):

$$\chi' = x17536428y \to 571x^r36428y \to 63x175428y \to 1x^r3675428y \to$$

$$\to 45763x128y \to 67543x128y \to 76543x128y \to 21x^r345678y \to$$
$$\to x12345678y.$$

Since $n = 8\,m$ then it takes n prefix–reversals for such moves. Thus, in a total of $19\,n/16$ prefix–reversals one can produce the identity permutation starting from χ, and the upper bound is established: $f(\chi) \leqslant 19\,n/16$.

For the lower bound, let

$$\chi \equiv \chi_0 \to \chi_1 \to \chi_2 \to \ldots \to \chi_{f(\chi)} \equiv I_n \tag{3.5}$$

be an optimal sequence of moves for χ. Each of χ_i for any $i = 1, \ldots, f(\chi)$ is called a *move*. Let us call a move *k–stable*, if it contains a substring of the form $1_k7_k\sigma 2_k8_k$ (or its reverse), where σ is a permutation of $\{3_k, 4_k, 5_k, 6_k\}$. We say that χ_i is an *event*, if χ_{i-1} is k–stable for some k but $\chi_i, \chi_{i+1}, \ldots \chi_{f(\chi)}$ are not.

Claim 1. *There are exactly m events in (3.5).*

To prove Claim 1, we notice that χ_0 is k–stable for $k = 1, \ldots, m$, and $\chi_{f(\chi)}$ is not k–stable for any k. Furthermore, no permutation can stop being k_1–stable and k_2–stable, $k_1 \neq k_2$, in only one move. □

Let us call χ_i a *waste* if χ_i has no more adjacencies than χ_{i-1}. (Here, by an adjacency in σ we mean any pair $(i, i+1)$ such that either $1 \leqslant i < n$ and $|\sigma_i - \sigma_{i+1}| = 1$, or $\sigma_1 = 1$, $\sigma_n = n$). Let w denote the total number of wastes among $\{\chi_i : i = 1, \ldots, f(\chi)\}$.

Claim 2. $n + w \leqslant f(\chi)$.

To see why this is true, one just has to notice that χ has no adjacencies, the identity permutation has n adjacencies, and any move that is not a waste creates just one adjacency. □

By Claim 1 one can conclude that in the optimal sequence that we are considering (3.5) there are m events as shown below:

$$\chi_{i_1} \overset{*}{\to} \chi_{i_2} \overset{*}{\to} \chi_{i_3} \overset{*}{\to} \ldots \overset{*}{\to} \chi_{i_m}, \tag{3.6}$$

where $\overset{*}{\to}$ is the transitive closure of $\to$, which means that we are interesting only in moves from an event to an event, and others moves are omitted.

Claim 3. *For all* $j, 1 \leqslant j \leqslant m-1$, *there exist a waste* χ_l *with* $i_j \leqslant l \leqslant i_{j+1}$.

To prove claim 3, suppose that it fails. In other words, suppose that there is an event i_j other than the last one, such that all moves χ_l, where $i_j \leqslant l \leqslant i_{j+1}$, construct a new adjacency without destroying an existing adjacency. Suppose that k is the appropriate index for which χ_{i_j-1} is the last k–stable permutation in the sequence considered. Then, $\chi_{i_j-1} = [x\, 1_k 7_k \sigma 2_k 8_k\, y]$, where x and y are strings of integers and σ is a permutation of $\{3_k, 4_k, 5_k, 6_k\}$. Notice that since our basic string $\tau = [17536428]$ is symmetric (in that $i + j = 9$ if and only if $\tau_i + \tau_j = 9$) this is not a loss of generality. For simplicity in our notation, we shall omit the subscript k in the rest of this part of our argument; we shall also assume that $\sigma = [5364]$, since the argument is identical for any σ). Thus, the last k–stable permutation in the sequence (3.6) is presented as $\chi_{i_j-1} = [x\, 17536428y]$. We distinguish among two cases.

Case 1. Let x be the empty string. Since χ_{i_j} is neither waste nor k–stable, we must have a permutation with a new adjacency:

$$\chi_{i_j} = [46357128y].$$

Now, we must not, according to our hypothesis, have a waste until *after* the next event. This, however, is impossible, since the first move after χ_{i_j}, which flips more than four elements is a waste.

Case 2. Let x be not the empty string. That is $\chi_{i_j-1} = [x17536428y]$. Since χ_{i_j} is neither a waste nor k–stable, it must be the case that $x = 9z$ and $\chi_{i_j} = [2463571z^R 98y]$. Again, we must not have a waste until after the next event. This means that the only moves permitted are local rearrangements of the integers $\{1, 2, 3, 4, 5, 6, 7\}$; thus

$$\chi_{i_j} = [2463571z^r 98y] \xrightarrow{*} [7654321z^r 98y].$$

Again, the next move has to be a waste. □

The theorem now follows directly from Claims 1, 2 and 3:

$$f(\chi) \geqslant n + w \geqslant n + \frac{m}{2} = 17\, n/16.$$

□

Thus, from this theorem we immediately have the following result.

Corollary 3.2.4 $17\, n/16 \leqslant diam(P_n)$ *for* n *a multiple of* 16.

3.2.4 Improved bounds by Heydari and Sudborough

Gates and Papadimitrou concluded in [33] that

> "... slightly better lower bounds may be conceivably proved by using different τ' – of length 7, say. However, we do not know how the upper and lower bounds can be narrowed significantly."

A lower bound was improved in 1997 by Heydari and Sudborough [40] on the basic permutation $\zeta = [1753642]$. Then as above for k, a positive integer, $\zeta_k = [1_k 7_k 5_k 3_k 6_k 4_k 2_k]$ where $m_k = m + 7(k-1)$. Consider the permutation

$$\varphi_n = \zeta_1 \, \zeta_2 \, \cdots \, \zeta_m, \tag{3.7}$$

where m is an even integer, $n = |\zeta| = 7\,m$, $\zeta = \zeta_1$.

Let $f(\varphi_n)$ be the number of prefix–reversals transforming a permutation φ_n to the identity permutation. Then the following theorem holds.

Theorem 3.2.5 [40] $\quad 15\,n/14 \leqslant f(\varphi_n)$ *for all* $n \equiv 0 \pmod{14}$.

The proof is similar to that given by Gates and Papadimitrou [33].

Proof. Let

$$\varphi \equiv \varphi_0 \to \varphi_1 \to \ldots \to \varphi_{f(\varphi_n)} \equiv I_n \tag{3.8}$$

be an optimal sequence of moves for φ_n. Each of φ_i, $i = 1, \ldots, f(\varphi_n)$, is called a *move*. A move is called *k–stable* if it contains a substring of the form

$$[1_k 7_k \sigma 2_k 8_k] = [1_k 7_k \sigma 2_k 1_{k+1}]$$

(or its reverse), where σ is a permutation of $\{3_k, 4_k, 5_k, 6_k\}$. (Assuming that $8_k = 1_{k+1}$, i.e., the $(n+1)$th pancake, is in its correct place at the bottom of the stack). We say that φ_i is an *event* if φ_{i-1} is k–stable, for some k, but $\varphi_i, \varphi_{i+1}, \ldots \varphi_{f(\varphi_n)}$ are not.

Claim 1. *There are exactly m events in (3.8).*

To prove Claim 1, we notice that φ_0 is k–stable for all $k = 1, \ldots, m$, and $\varphi_{f(\varphi_n)}$ is not k–stable for any k. Furthermore, no permutation can stop being k_1–stable and k_2–stable in only one move for $k_1 \neq k_2$. □

A *waste* is defined as above: φ_i is a waste if it has no more adjacencies than φ_{i-1}. The total number of wastes among $\{\varphi_i : i = 1, \ldots, f(\varphi_n)\}$ is denoted by w.

Claim 2. $n + w \leqslant f(\varphi_n)$.

This Claim is true because φ_n has no adjacencies, and the identity permutation I_n has n adjacencies, and any move that is not a waste creates just one adjacency. □

By Claim 1, it is concluded that in the optimal sequence we consider (3.8) there are m events as shown by:

$$\varphi_{i_1} \overset{*}{\to} \varphi_{i_2} \overset{*}{\to} \varphi_{i_3} \overset{*}{\to} \ldots \overset{*}{\to} \varphi_{i_m}, \tag{3.9}$$

where $\overset{*}{\to}$ is the transitive closure of $\to$, which means that we are interesting only in moves from an event to an event, and others moves are omitted.

Claim 3. *For all $j, 1 \leqslant j \leqslant m-1$, there exist a waste φ_l with $i_j \leqslant l \leqslant i_{j+1}$.*

To prove Claim 3, suppose that there is an event i_j other than the last one, such that all moves φ_l, $i_j \leqslant l \leqslant i_{j+1}$, construct a new adjacency without destroying an existing adjacency. Let k be the appropriate index for which φ_{i_j-1} is the last k–stable permutation in the sequence (3.8). Then

$$\varphi_{i_j-1} = [x\, 1_k 7_k \sigma 2_k 8_k\, y] = [x\, 1_k 7_k \sigma 2_k 1_{k+1}\, y],$$

where x and y are strings of integers and σ is a permutation of $\{3_k, 4_k, 5_k, 6_k\}$. Notice that because the string $\zeta = [1_k 7_k 5_k 3_k 6_k 4_k 2_k 8_k]$ is symmetric (in that $i + j = 9$ if and only if $i_k + j_k = 9 + 2(k-1)7$), this is without loss of generality. For simplicity, i_k is replaced just by i in the rest of the argument; it is also assumed that $\sigma = [5364]$ because the argument is identical for any σ. Thus, the last k–stable permutation in the sequence (3.8) is presented as:

$$\varphi_{i_j-1} = [x\, 17536428y].$$

We distinguish two cases.

Case 1. Let x be the empty string. Whereas φ_{i_j} is neither a waste nor k–stable, we must have a permutation with a new adjacency:

$$\varphi_{i_j} = [46357128y].$$

Now, we must not, according our hypothesis, have a waste until *after* the next event. This, however, is impossible, because the first move after φ_{i_j} which flips more that four elements is a waste. Note that $8_k = 1_{k+1}$ remains fixed in any prefix–reversal of the four elements, and therefore all such prefix–reversals do not change whether φ_{i_j-1} contains (or does not contain) the substring $[1_{k+1}7_{k+1}\sigma 2_{k+1}8_{k+1}]$ (or its reversal) for any σ. Also, whereas $1_k = 8_{k-1}$ is the first element in φ_{i_j-1} and therefore quite apparently is not next to 2_{k-1}, the move φ_{i_j-1} does not contain the substring $[8_{k-1}2_{k-1}\sigma 7_{k-1}1_{k-1}]$ (or its reversal) for any σ. Whereas any prefix–reversal of the first four elements will not create such a substring and break it, there is no event created by such prefix–reversals that involves $(k-1)$–stability or $(k+1)$–stability.

Case 2. Let x be not the empty substring. That is, $\varphi_{i_j-1} = [x17546428y]$. Since φ_{i_j} is neither a waste nor k–stable, it must be the case that $x = 9z$ for some z and $\varphi_{i_j} = [2463571z^R98y]$. Again, we must not have a waste until after the next event. This means the only moves permitted are local rearrangements of the integers $\{1,2,3,4,5,6,7\}$; thus

$$\varphi_{i_j} = [2463571z^R98y] \xrightarrow{*} [7654321z^r98y].$$

Again, the next move has to be a waste. Suppose that φ_{i_j} contains a substring of the form $[1_{k-1}7_{k-1}\sigma 2_{k-1}8_{k-1}]$ for some σ. Recall that $8_{k-1} = 1_k$, so each move involving a local permutation of the integers $\{1,2,3,4,5,6,7\}$, including the move $\varphi_{i_j} = [2463571z^r98y]$, does not change the status of $(k-1)$–stability. Similarly, for $(k+1)$–stability, because the symbol $8_k = 1_{k+1}$ remains fixed throughout. □

Theorem 3.2.5 now follows directly from Claims 1, 2 and 3:

$$f(\varphi_n) \geqslant n + w \geqslant n + \frac{m}{2} = 15\,n/14.$$

□

Corollary 3.2.6 $15\,n/14 \leqslant diam(P_n)$ *for* n *a multiple of* 14.

Recently an improved upper bound was presented by Hal Sudborough in cooperation with a team from University of Texas at Dallas (see [21]) as follows:

$$diam(P_n) \leqslant 18n/11.$$

3.2.5 Exact values on the diameter of the Pancake graph

Heydari and Sudborough also computed the diameter of the Pancake graph P_n up to $n = 13$. The diameter for $n = 14, 15$ was found in 2005 by Yuusuke Kounoike, Keiichi Kaneko, and Yuji Shinano [56]. In one year, in 2006 the same authors and Shogo Asai [4] showed the diameter for $n = 16, 17$. In 2011 the diameter for $n = 18, 19$ was found by Cibulka [19].

The table of the values for the diameter $d = diam(P_n)$ of the Pancake graph P_n, where $2 \leqslant n \leqslant 19$, is presented below:

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
d	1	3	4	5	7	8	9	10	11	13	14	15	16	17	18	19	20	22

3.3 Burnt Pancake problem

The *Burnt Pancake problem* was introduced by Gates and Papadimitrou in [33]. This problem concerns sorting a stack of pancakes not only of different sizes but each with side burnt. Initially, the pancakes are arbitrary ordered and each pancake may have either side up. After sorting, the pancakes must not only be in size order, but must have their burnt sides face down. Two–sided pancakes can be represented by a signed permutation on n elements with some elements negated.

So, in this case we deal with the *hyperoctahedral group* B_n, which is defined as the group of all permutations π^σ acting on the set $\{\pm 1, \ldots, \pm n\}$ such that $\pi^\sigma(-i) = -\pi^\sigma(i)$ for all $i \in \{1, \ldots, n\}$. An element of B_n is a *signed permutation*, i.e. a permutation with a sign attached to every entry and determined by two pieces of information: $|\pi(|i|)|$, which permutes $\{1, \ldots, n\}$, and the sign of $\pi^\sigma(i)$ for $1 \leqslant i \leqslant n$. This gives a bijection between B_n and the wreath product $\mathbb{Z}_2 \wr Sym_n$ of the "sign–change" cyclic group $\mathbb{Z}_2$ with the symmetric group Sym_n; thus $|B_n| = 2^n n!$.

The Burnt Pancake problem consists of finding the diameter of the Cayley graph $BP_n,\ n \geqslant 2$, which is defined on the hyperoctahedral group B_n of signed permutations and generated by *sign–change prefix–reversals* r_i^σ,

$1 \leqslant i \leqslant n$, changing signs of reversing elements from any substring $[1, i]$, of a signed permutation π^σ when multiplied on the right, i.e.

$$[\pi_1 \dots \overline{\pi}_j \dots \pi_i \dots \pi_n] r_i^\sigma = [\overline{\pi}_i \dots \pi_j \dots \overline{\pi}_1 \dots \pi_n].$$

Let us note that the sign–change prefix–reversals r_1^σ just changes a sign of the first element of permutation. The Cayley graph defined above is called *Burnt Pancake graph* [52].

There is the following open problem.

Problem 3.3.1 *What is the diameter of the Burnt Pancake graph?*

Gates and Papadimitrou [33] gave the following bounds on the diameter of the Burnt Pancake graph:

$$3n/2 - 1 \leqslant diam(BP_n) \leqslant 2n + 3.$$

The improved bounds for the diameter $diam(BP_n)$ of the Burnt Pancake graph were given by Cohen and Blum [23]:

$$3n/2 \leqslant diam(BP_n) \leqslant 2n - 2,$$

where the upper bound holds for $n \geqslant 10$.

It was also conjectured that the worst case for sorting signed permutations (burnt pancakes) is the negative identity permutation $I_n^- = [-1 - 2 \dots - n]$. Later Hyedari and Sudborough [40] showed that if the conjecture is true then the diameter of the Burnt Pancake graph is

$$diam(BP_n) \leqslant 3(n+1)/2,$$

since I_n^- can be sorted in $3(n+1)/2$ steps for all $n = 3 \pmod 4$ and $n \geqslant 23$. Currently, exact values of $d = diam(BP_n)$ are known for $n \leqslant 18$ and presented as follows:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
d	1	4	6	8	10	12	14	15	17	18	19	21	22	23	24	26	28	29

3.4 Sorting by reversals

In general, a pancake stack is an example of a data structure. In computer science and molecular biology the problem presented above is also related to sorting by prefix–reversals. The Pancake graph has practical applications in parallel processing since it corresponds to the n–dimensional pancake network such that this network has processors labeled with each of the $n!$ distinct permutations of length n. Two processors are connected when the label of one is obtained from the other by some prefix–reversal. The diameter of this network corresponds to the worst communication delay for transmitting information in a system. The pancake sorting can also provide an effective routing algorithm between processors. There is a very nice survey by Marie–Claude Heydemann [39] about Cayley graphs as interconnection networks, which can be recommended for more details.

Recent advances in genome identification have also brought to light questions in molecular biology very similar to the Pancake problem. Differences in genomes are usually explained by accumulated differences built up in the genetic material due to random mutation and random mating. Another mechanism of evolution was discovered in 1986 by Jeffrey D. Palmer and Laura A. Herbon [67]. Comparing two genomes one can often find that these two genomes contain the same set of genes. But the order of the genes is different in different genomes. For example, it was found that both human X chromosome and mouse X chromosome contain eight genes that are identical. In human, the genes are ordered as $[4\,6\,1\,7\,2\,3\,5\,8]$ and in mouse, they are ordered as $[1\,2\,3\,4\,5\,6\,7\,8]$. It was also found that a set of genes in cabbage are ordered as $[1\ -5\,4\ -3\,2]$ and in turnip, they are ordered as $[1\,2\,3\,4\,5]$. The comparison of two genomes is significant because it provides us some insight as to how far away genetically these species are. If two genomes are similar to each other, they are genetically close. This has inspired some molecular biologists to look at the mechanisms which might shuffle the order of the genetic material. One way of doing this is the prefix–reversals or just reversals. Analyzing the transformation from one species to another is analogous to the problem of finding the shortest series of reversals to transform one into the other.

In the 1980's it was shown that the difference in genomes is also ex-

plained by a small number of reversals which are the operations reversing the order of a substring of a permutation. Reversal distance measures the amount of evolution that must have taken place at the chromosome level, assuming evolution proceeded by inversion. More precisely, a reversal $r_{i,j}$ is the operation of reversing segments $[i, j]$, $1 \leqslant i < j \leqslant n$, of a permutation when multiplied on the right, i.e. $[\ldots \pi_i \, \pi_{i+1} \ldots \pi_{j-1} \, \pi_j \ldots] r_{i,j} = [\ldots \pi_j \, \pi_{j-1} \ldots \pi_{i+1} \, \pi_i \ldots]$. The reversal distance $d(\pi, \tau)$ between two permutations π and τ is the least number d of reversals needed to transform π into τ, i.e., $\pi r_{i_1,j_1} \ldots r_{i_d,j_d} = \tau$. The corresponding Cayley graph defined on the symmetric group and generated by the reversals is called the Reversal graph $Sym_n(R)$. It was defined in Section 1.3.5 where some its properties were also presented.

The problem of determining the smallest number of reversals required to transform a given permutation into the identity permutation is called *sorting by reversals*. Mathematical analysis of the problem was initiated by David Sankoff [73] in 1992, and then continued by another authors. There are two algorithmic subproblems.

The first one is to find the reversal distance $d(\tau_1, \tau_2)$ between two permutations τ_1 and τ_2. Notice that the reversal distance between τ_1 and τ_2 is equal to the reversal distance between $\pi = \tau_2^{-1}\tau_1$ and the identity permutation I_n. It was shown by John Kececioglu and David Sankoff [46] in 1995 and by Vineet Bafna and Pavel Pevzner [11] in 1996 that

$$\max_{\pi \in Sym_n} d(\pi, I_n) = n - 1.$$

The path distance in the Reversal graph $Sym_n(R)$ corresponds to the reversal distance between two permutations. Hence, its diameter is $n - 1$, and the only permutations needing these many reversals are the Gollan permutation γ_n and its inverse, where the Gollan permutation, in one–line notation, is defined as follows

$$\gamma_n = \begin{cases} [315274 \ldots n-3 \, n-5 \, n-1 \, n-4 \, n \, n-2], & \text{if } n \text{ is even} \\ [315274 \ldots n-6 \, n-2 \, n-5 \, n \, n-3 \, n-1], & \text{if } n \text{ is odd.} \end{cases}$$

As it was shown above, the signed permutations are also used to represent genomes when a direction of genes is significant. The *reversal distance* $\rho(\pi^\sigma, \tau^\sigma)$ between two signed permutations π^σ and τ^σ is the least

number ρ of sign–change reversals needed to transform π^σ into τ^σ, i.e., $\pi^\sigma r^\sigma_{i_1,j_1} \dots r^\sigma_{i_d,j_d} = \tau^\sigma$. The *Reversal graph* $B_n(R^\sigma)$ is defined on B_n and generated by the sign–change reversals from the set $R^\sigma = \{r^\sigma_{i,j} \in B_n, 1 \leqslant i \leqslant j \leqslant n\}$, where a sign–change reversal $r^\sigma_{i,j}$, $1 \leqslant i \leqslant j \leqslant n$, is defined as the operation of reversing segments $[i, j]$ of a signed permutation π^σ with flipping the signs of its elements, e.g.

$$[\dots \pi_i \, \overline{\pi}_{i+1} \dots \pi_{j-1} \, \pi_j \dots] r^\sigma_{i,j} = [\dots \overline{\pi}_j \, \overline{\pi}_{j-1} \dots \pi_{i+1} \, \overline{\pi}_i \dots].$$

In the case of sorting by sign–change reversals, we have to find the reversal distance $\rho(\tau^\sigma_1, \tau^\sigma_2)$ between two signed permutations τ^σ_1 and τ^σ_2, or between $\pi^\sigma = (\tau^\sigma_2)^{-1}\tau^\sigma_1$ and the positive identity permutation defined as $I^+_n = [+1 \dots +n]$. It was shown by Knuth [58] in 1994 (see Exercise 5.1.4–43) that at most $n+1$ sign–change reversals are needed to sort any signed permutation to the positive identity permutation, for all $n > 3$, i.e.

$$\max_{\pi^\sigma \in B_n} \rho(\pi^\sigma, I) = n + 1.$$

The path distance in the Reversal graph $B_n(R^\sigma)$ corresponds to the reversal distance between two signed permutations. This means that its diameter is $n + 1$ and the following permutations, written in one–line notation, are at this maximum distance from the identity permutation I^+_n:

$$\pi^\sigma = \begin{cases} [+n \; +(n-1) \dots +1], & \text{if } n \text{ is even,} \\ [+2 \; +1 \; +3 \; +n \; +(n-1) \dots +4], & \text{if } n > 3 \text{ is odd.} \end{cases}$$

In 2001, it was also shown by David Bader, Bernard Moret, and Mi Yan [10] that the reversal distance could be calculated in linear time for signed permutations.

The next subproblem here is how to reconstruct a sequence of reversals which realizes the distance. Its solutions are far from unique. In 1994 it was shown by John Kececioglu and David Sankoff [45] that the problem is NP–hard for unsigned permutations. However, it is polynomial for signed permutations as it was shown by Sridhar Hannenhalli and Pavel Pevzner [38] in 1999. The 1.5–approximation algorithm for sorting unsigned permutations was presented by David Christie [22] in 1998. One of the most effective algorithms that sort signed permutations by reversals was presented by Haim Kaplan and Elad Verbin [44] in 2003.

An interesting problem related to sorting by reversals is the problem of sorting by transpositions of long fragments which was considered by Vineet Bafna and Pavel Pevzner [12] in 1998. Lower bounds on transposition distance between permutations are found and approximation algorithms for sorting by transpositions are presented in this paper. Some open problems in genome rearrangements are also discussed there. A natural generalization of sorting by transpositions is the problem of sorting by block–interchanges which was considered by David Christie in his thesis "Genome rearrangement problems" in 1998. In particular, he proved that this problem in NP–hard. As one can see, genome rearrangement problems have proved so interesting from a combinatorial point of view that the field now belongs as much to mathematics as to biology.

One more interesting problem related to sorting by reversals is the problem of sorting permutations by the fixed–length reversals (k–reversals). This problem is implicit in the popular puzzle $TOP - SPIN^{TM}$ which consists of a permutation of 20 numbered disks on an oval track, with a turnstile capable of reversing a string of 4 consecutive disks. The goal is to sort the disks to the identity permutation using reversals. The more general problem, with permutations of n disks and a turnstile of size k was considered and solved by Ting Chen and Steven Skiena [20]. Note that sometimes it is impossible to sort permutation using only reversals of certain length. For example, an odd length reversal does not change the parity of the position of any element it acts on, so odd length reversals cannot sort any permutation of the form $\pi = [2\,1\,\ldots\,n]$. The authors gave a complete description for all n and k of how many permutations of length n can be sorted using only reversals of length k. In particular, it was shown that $O(n^{3/2})$ k-reversals suffice to transform any permutation to the identity permutation when $k \approx \sqrt{n}$. The number of connected components on the corresponding Cayley graphs generated by the fixed–length reversals was also discussed there. For instance, in the trivial case of n-reversals, any permutation can be transformed only to its inverse permutation. Thus, there are $n!/2$ connected components for the symmetric group Sym_n with $k = n > 2$.

Chapter 4

Further reading

Classical problems on Cayley graphs such as classification, isomorphism and enumeration of Cayley graphs were reviewed by Ming–Yao Xu [80] in 1996 and Cai Heng Li [62] in 2002. The last article is devoted to surveying results, open problems and methods based on deep group theory, including the finite simple group classification, and on combinatorial techniques. It contains 121 references on the topics. Very good survey articles on the classification problems were written by Laszlo Babai [5] in 1981 as well as presented in the book on "Topics in Algebraic Graph Theory" published in 2004 [13]. Isomorphisms of Cayley graphs were considered in the older classic paper by Laszlo Babai and Peter Frankl [8] in 1978 and in the modern handbook by Laszlo Babai [6] in 1996. Much more on the combinatorics and groups, including the useful list of references on this topic, is given in Peter Cameron's IPM Lecture Notes [18] on "Combinatorics and Groups" published in 2004 and in Adalbert Kerber's book [47] on "Algebraic Combinatorics via Finite Group Actions" published in 1991. The questions concerning eigenvalues, expanders and random walks in Cayley graphs (which not mentioned in these notes) were considered by Alex Lubotzky [64] in 1995.

There are survey papers on hamiltonian problem for graphs, Cayley graphs and digraphs, presented by Dave Witte and Joseph Gallian [79] in 1984, Ronald Gould [37] in 1991 and Stephen Curran and Joseph Gallian [25] in 1996. In the last paper the main results are chronicled and some open problems and conjectures are included. These surveys also contain some material on related topics such as hamiltonian decompositions,

hamiltonian–connected and pancyclic graphs and digraphs, as well as an extensive bibliography of papers in the area. A short survey of various results in hamiltonicity of Cayley graphs could be also found in the paper by Igor Pak and Radoš Radoičić [68].

To get more information about Cayley graphs in computer science, I would like to recommend two surveys by Sheldon Akers and Balakrishnan Krishnamurthy [1], and S. Lakshmivarahan, Jung–Sing Jwo and Sudarshan Dhall [59] written in 1989 and 1993, correspondingly. In these papers some properties of Cayley graphs such as edge–transitivity (line symmetry), hierarchical structure (allowing recursive construction), high fault tolerance and so on, are discussed. Another good reference on this subject is the report by Marie–Claude Heydemann [39] on "Cayley Graphs as Interconnection Networks" where routing problems including connectivity, diameter and loads of routing are considered for Cayley graphs and for some generalizations of Cayley graphs.

Some combinatorial problems arising in computational molecular biology have connections with classical combinatorial problems on Cayley graphs. For more details on such applied problems, I recommend the books by Pavel Pevzner [69] written in 2000 and the book on "Current Topics in Computational Molecular Biology" [74] with a good chapter on "Genome rearrangement" by David Sankoff and Nadia El–Mabrouk. There is also a very nice course "Algorithms for Molecular Biology" by Ron Shamir presented at his homepage [75]. One more recently published book could be also recommended for reading. This is "Combinatorics of Genome Rearrangements" by Guillaume Fertin, Anthony Labarre, Irena Rusu, Eric Tannier and Stephane Vialette [31] published in 2009.

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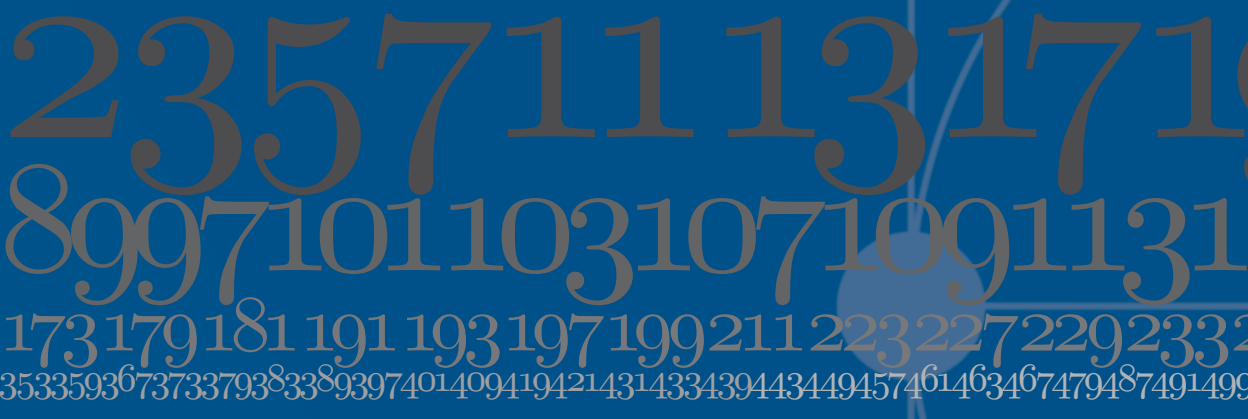

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ISBN 978-961-6832-50-2