The Equals Sign: The Challenges of Learning Arithmetic

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The authors of the paper draw attention to the equals sign as a key element in learning arithmetic and the basis for learning other areas of mathematics, and point out that students, as a rule, understand this concept operationally, completely lacking relational understanding of it. The aim of the organized research was to examine how first-grade and third-grade students understand the concept of the equals sign at operationalized levels (operational, basic relational, and complex relational), and whether there are any differences between them. Testing first-grade and third-grade students in primary school ($N = 212$) yielded results that show that operational understanding of the equals sign is prevalent among students, that relational understanding is underdeveloped, and that third-grade students demonstrate better relational understanding of the concept than first-graders. In consequence, the authors propose that developing relational understanding of the equals sign should be given more attention in arithmetic classes.

Keywords: word problems, keywords, diagram, schema-based instruction, primary school

Introduction

Systemic mathematics education begins with arithmetic. From the very first grade of primary school, students begin to form the concepts of natural numbers and operations with them. Among the numerous concepts and symbols that students learn, the equals sign is the most scrutinized by mathematics education researchers. Numerous authors point out that the equals sign is one of the most important concepts and symbols in mathematics (Darr 2003; Milinković, Maričić, and Đokić 2022; Milinković, Maričić, and Lazić

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2022; Wahyuni and Herman, 2020; Carpenter et al. 2003; McNeil et al. 2006; Stephens et al. 2013), and one of the very foundations for learning mathematics, because there is no branch of mathematics that does not use it (Carpenter et al. 2003; McNeil et al. 2006); especially, it is the essential concept for algebra achievement (Van Amerom 2003; Chimoni, Pitta-Pantazi, and Christou 2018). It is particularly emphasized that the failure to grasp the meaning of the equals sign in early grades can be a cause of learning difficulties later in life, even at a higher level of education (Baiduri 2015; Knuth et al. 2006).

However, the authors also agree that, in addition to being an important concept for understanding mathematics, understanding the equals sign is very challenging for most students at all levels of mathematics education (Alibali et al. 2007; Kieran 1981) and that there are numerous problems plaguing its correct understanding (Carpenter and Levi 2000; Filloy and Rojano 1989; Freiman and Lee 2004; Kieran 1981; Knuth et al. 2006; McNeil and Alibali 2005; Sfard and Linchevski 1994), which can later be reflected in the learning of other mathematical content, especially algebra. 'The difficulties related to the misunderstanding of the equals sign in algebra stem from the precarious foundations on which said concept is formed in arithmetic learning' (Milinković, Maričić, and Đokić 2022, 27). As pointed out by Ardiansari, Suryadi, and Dasari, all the above shows that 'understanding the equal sign has become the focal point of reform and research efforts in mathematics education as a proven strong and long-term "problem" that can be experienced by elementary, middle school and even college students' (Ardiansari, Suryadi, and Dasari 2022, 95).

In the light of the considerations stated above, the paper will focus on examining students' understanding of the equals sign.

Understanding the Concept of the Equals Sign

Mathematical equivalence is the relation between two quantities that are equal and interchangeable (McNeil et al. 2015; Kieran 1981). The concept of the equals sign implies the unity of symbolic notation and the meaning expressed through equality and equivalence. Numerous studies show that the majority of students do not fully understand the equals sign and that there are numerous problems that arise with regard to the equals sign and their ability to understand it (Booth 1988; Kieran 1981; Sfard and Linchevski 1994, Filloy and Rojano 1989; Carpenter and Levi 2000;). Research shows that students view and understand the equals sign as a result of arithmetical operation rather than mathematical equality (Kieran 1981; Stephens et al. 2013; Vermeulen and Meyer 2017). Falkner, Levi, and Carpenter (1999) showed that most students perceive the meaning of the equals sign as an operational symbol, calculating the number from the left-hand side to the right. The reason for this understanding of the equals sign, or lack thereof, among students arises from the manner in which this concept is learned in arithmetic. Students first encounter this concept in arithmetic, usually in situations where there is an expression on the left followed by an equals sign, and they are expected to calculate, i.e. determine its value. Consequently, research shows that students largely view the equals sign as the command 'to calculate' or 'to compute' (Baroody and Ginsburg 1983; Behr, Erlwanger, and Nichols 1980; Cobb 1987; Ilič and Zeljič 2017; Kieran 1981; McNeil and Alibali 2005), i.e. they interpret it as a command to perform arithmetic calculations, or 'see "=" as an instruction to complete an operation' (Parslow-Williams and Cockburn 2008, 36). Kieran (1981) also draws attention to the fact that the use of the equals sign in arithmetic classes steers students toward a wrong conceptualization of this concept. It is crucial that students develop and understand the equals sign both *operationally* in terms of performing certain operations, i.e. commands 'to calculate' and 'to compute' and *relationally,* in terms of equivalence of the left and the right side of the equality, at an early age (Kieran 1981; Knuth et al. 2006; McNeil et al. 2006).

The difficulties reflected in mathematics education, and referring to a narrow understanding of the equals sign, i.e. in operational but not in relational terms, stem from the fact that children are first introduced to arithmetic, at the very outset of their mathematics education. The students' understanding of the equals sign in the operational sense is related to the habit that the expression is always located on the left side of the equality, while its results are located on the right. Let us take $5 + 6 = \_\_\_$ as an example, because it possesses a typical arithmetic interpretation of the equals sign. In the given example, students naturally feel the need to calculate the sum of 5 and 6, and write the number 11 in the blank. However, when they are presented with the following problem: $5 + 6 = \_\_\_ + 2$, they must understand the equivalence between the left and the right side of the equality in order to successfully solve the given problem. Research shows that using problems that deviate from the standard form of equality in arithmetic, which implies that the expression is always on the left side of the equality (for example, $\_\_\_ = 5 + 6$), increases the likelihood of developing relational understanding of the equals sign (McNeil and Alibali 2005; McNeil et al. 2011). Solving problems with a non-standard format helps students improve their understanding of the equals sign, especially if that process involves the use of words that refer to the primary meaning of the equals sign, such as 'equals to' or 'is the same as' (Rittle-

Johnson and Alibali 1999), or the expressions 'the left and the right side of the equals sign have the same meaning' and 'same as' (Knuth et al. 2006). In addition to the concept of sameness in the relational sense, researchers also propose using expressions that signify interchangeability instead of sameness, such as 'the left and the right side are interchangeable' (Jones et al. 2012; Jones et al. 2013). The mathematical meaning of the equals sign referring to interchangeability shows an analogy in the students' cognitive understanding, and a more sophisticated interpretation of this concept.

Although operational understanding may be sufficient in solving classical elementary arithmetic problems, misconceptions about the meaning of the equals sign can later lead to problems with solving equations (Alibali et al. 2007; Carpenter et al. 2003; Prediger 2009). Developing an understanding of the symmetrical nature of equality requires one to grasp the fact that a mathematical symbol fundamentally expresses a relationship between quantities, and that it should not be viewed merely as a signal for performing mathematical operations.

The ultimate goal that is being pursued is that the students' understanding of the concept of the equals sign ends at the relational level, i.e. the level that expresses equivalence. According to Byrd et al. (2015), without relational understanding, the algebraic principle of maintaining equality is meaningless, wherein children are left to struggle and memorize countless arbitrary rules in equation transformations. In order for children to understand the equals sign as 'it is equivalent' or 'it is the same as,' they first need to understand expressions and equalities as a whole, i.e. understand the equals sign relationally. Thus, research by Molina and Ambrose (2008) confirms that the expression or equation must be observed as an entity of its own. Only through the unity of the elements in the expression is it possible to influence the correct notions about the equals sign, and thus, about the correct transformation of the equation in its solution.

The solution to this problem requires comprehension 'that the equals sign lies in the deeper understanding of this concept and developing an understanding of this concept at the relational, instead of just the operational level' (Milinković, Maričić, and Lazić 2022, 98). Accomplishing this task is a complex process for teachers, so it must be systematic and gradual. A more detailed operationalization of the levels of understanding of the equals sign proposed by the following authors (Jones et al. 2013; Lee and Pang 2020; McAuliffe et al. 2020; Milinković, Maričić, and Lazić 2022; Rittle-Johnson et al. 2011) can be extremely helpful in this process. The levels of understanding of the equals sign are shown in table 1.

Table 1 Levels of Understanding of the Equals Sign

Level of understanding	Expected outcomes
Level 4: Real relational	Student understands equivalence in real-world context problems.
Level 3: Complex relational	Student understands equivalence in complex equalities that feature multiple equals signs.
Level 2: Basic relational	Student understands the equals sign as a symbol of equivalence in equalities that feature expressions on both sides of the equality. Student uses relational thinking and understands equivalence in simple equalities.
Level 1: Operational	Student understands the equals sign as a command 'to calculate.' Student understands simple equalities that feature expressions on both sides of the equals sign.

Notes Adapted from Milinković, Maričić, and Lazić (2022, 98).

Level 1 implies operational understanding of equality where the student views the equals sign solely in the function of performing a calculation, i.e. as the command 'to calculate.' *Level 2* implies a somewhat more advanced understanding of the equals sign. The development of this level reflects the student's ability to understand equivalence, and not understand the equals sign solely in an operational sense. At this level, the student views the equals sign as a mathematical equivalence, in other words, as a sign that signifies the equivalence of the left and the right side of the equation. *Level 3* involves students' understanding of complex equalities in which the equals sign occurs repeatedly (Lee and Pang 2020; Milinković, Maričić, and Lazić 2022). Understanding the equals sign at a complex relational level involves examples of equations where the equals sign occurs repeatedly as a link between expressions (e.g. $2 + 3 = \_\_\_ - 1 = 9 - \_\_\_ = \_\_\_$). *Level 4* involves understanding of the equals sign in the context of real world problem solving. 'This involves situations in which students are expected to solve specific problems using the balance method, i.e. jumping from one side of the equation to the other' (Milinković, Maričić, and Lazić 2022, 99).

All individual levels of understanding are interconnected, but it should be considered 'that the continuous nature of the model means that the levels should not be interpreted as discrete stages' (Rittle-Johnson et al. 2011, 3). The development of one level of understanding does not exclude the development of a different level, nor does the development of one level cancel another. The concept of the equals sign formed at an operational level can be gradually transformed into a relational meaning, which is what arithmetic classes aim to accomplish, and which will later lay the foundations for learning more abstract content (Milinković, Maričić, and Lazić 2022). The above confirms that it is necessary to examine the way in which junior primary school students understand the equals sign, and whether there are differences in the understanding of this concept between students who have just embarked on their mathematical journey, and third-graders.

Research Methodology

The research was conducted on a sample of 212 first-grade and third-grade students of two primary schools in Užice (Republic of Serbia) during 2022/2023. The sample was chosen through random sampling, and included 109 first-graders and 103 third-graders. The reason we selected students of the first grade of primary school is because we wanted to examine how students perceive the equals sign, given that they only get to use it in arithmetic at this educational stage. On the other hand, we decided to test third-graders because by that stage, they already possess significant experience with the equals sign, as well as with arithmetic and algebra. In this way, we wanted to compare student understanding of the equals sign depending on their experience with mathematical content, and with regard to the concept of the equals sign.

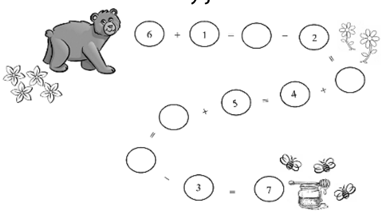
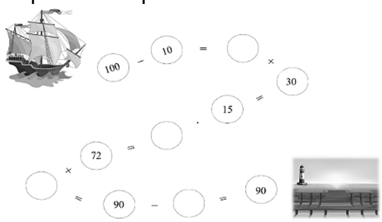
The aim of the paper was to examine how students of the first and the third grade of primary school understand the concept of the equals sign with regard to operationalized levels, and whether there are any differences in understanding between students of different ages. We decided to examine the development of the equals sign concept at the following three levels: *operational, basic relational,* and *complex relational.*

The following research tasks were defined in relation to the goal:

1. Investigate the development of the operational level of understanding of the equals sign between first-grade and third-grade students in primary school;
2. Investigate the development of the basic relational level of understanding of the equals sign between first-grade and third-grade students in primary school;
3. Investigate the development of the complex relational level of understanding of the equals sign between first-grade and third-grade students in primary school.

In order to investigate the given tasks, we used testing, designing two tests for that purpose – one for first-grade students (T1), and another for third-

Table 2 Test Tasks by Level of Understanding of the Equals Sign

(1)	First grade	Third grade	(2)
1	Write the correct number in the blanks: 3 + 1 = ___ ___ = 7 + 1 2 + 6 = ___ ___ = 4 + 3	Write the correct number in the blanks: 300 + 100 = ___ ___ = 440 + 30 230 + 160 = ___ ___ = 580 + 80	Operational level
2	Write the missing number in the blanks: ___ + 2 = 5 3 + ___ = 6 9 = ___ + 5 10 = 8 + ___	Write the missing number in the blanks: ___ + 220 = 550 35 + ___ = 150 999 = ___ + 1 470 = 30 + ___	
3	Write the missing number in the blanks: 2 + 6 = ___ + 5 10 − 8 = 1 + ___ ___ + 1 = 7 + 3 5 + ___ = 9 − 2 5 + ___ = 4 + 1 8 − 4 = ___ − 2	Write the missing number in the blanks: 200 + 65 = ___ + 5 1000 − 800 = 100 + ___ ___ + 10 = 75 + 35 55 + ___ = 90 − 20 500 + ___ = 400 + 100 840 − 40 = ___ − 20	Basic relational level
4	Write the missing number in the blanks: 9 + 1 = ___ + 7 = ___ + 5 9 − ___ = ___ + 1 = 3 4 + 1 = 10 − ___ = 2 + ___ 10 = 8 + ___ = 6 + ___	Write the missing number in the blanks: 16 + 44 = 100 − ___ = ___ + 30 50 + ___ = 200 − 50 = ___ + 30 400 − ___ = 110 + 90 = ___ 300 = ___ − 200 = 200 + ___	Complex relational level
5	Write the missing numbers and help the bear reach the honey jar: 	Write the missing numbers and help the ship reach the pier: 	

Notes Column headings are as follows: (1) task, (2) Level of understanding of the equal sign.

grade students (T2). Each test included five tasks (problems) for testing the development of the equals sign in their respective grades. The tests were identical in terms of requirements; only the blocks of numbers that formed the framework for the tasks were different. Each task consisted of several examples designed so as to match different levels of understanding of the equals sign (table 2).

In order to examine the understanding of the equals sign at *operational level,* we used the first and the second task, the aim of which was to determine whether students view the equals sign as a command 'to calculate,' or relationally. The third task aimed to examine the development of understanding of the equals sign at *basic relational level.* The fourth and the fifth task re-

Table 3 Cronbach Alpha Coefficient

Item	First grade	Third grade
Cronbach's Alpha	0.812	0.815
Cronbach's Alpha Based on Standardized Items	0.786	0.798
N of items	5	5

ferred to the complex relational level, which included examples of equalities comprising multiple equals signs.

The students had 45 minutes (one school period) to complete the test. The accuracy of the tasks depended on their success in solving them, so a task was *correct* if all given examples were correctly solved, *partially correct* if only a portion of the examples were correctly solved, and *incorrect* if none of the examples was correctly solved. The data obtained in the test were processed qualitatively and quantitatively, and the research results were converted into percentages and presented in tables. Quantitative analysis included the analysis of the results converted into percentages, whereas qualitative analysis relied on the analysis of typical errors that occurred in the process of solving tasks.

Test reliability was determined using Cronbach's alpha. The value of Cronbach's alpha (table 3) was 0.812 on the test for first-graders (T1), and 0.815 on the test for third-graders (T2), demonstrating high reliability and internal consistency of the scale for this sample size.

A chi-square test was used to determine the statistical significance of the differences in results scored by students of different ages.

Results and Discussion

Understanding the Equals Sign at Operational Level

With the first research task, we aimed to examine whether students understand the equals sign at operational level, i.e. whether they view it as a command 'to calculate.' The students were presented with two tasks. The first task required students to calculate the value of an expression located only on one side of the equals sign. The majority of students from both grades successfully solved this task – the completion rate (95.41%) among first-graders and (94.17%) among third-graders (table 4). Differences in performance were almost non-existent, as confirmed by the results of the chi-square test (χ^2 = 0.165, $df = 1$, $p = 0.685$).

The majority of partially successful students made errors in one out of four examples, and none of the students were completely unsuccessful, i.e. there

Table 4 Development of the Equals Sign at Operational Level

Task/grade		(1)		(2)		(3)		Chi-square
		f	*f*%	*f*	*f*%	*f*	*f*%	
1	First	104	95.41	5	4.59	0	0.00	$\chi^2 = 0.165$, $df = 1$, $p = 0.685$
	Third	97	94.17	6	5.83	0	0.00	
2	First	93	85.32	15	13.76	1	0.92	$\chi^2 = 5.292$, $df = 2$, $p = 0.071$
	Third	77	74.76	26	25.24	0	0.00	

Notes Column headings are as follows: (1) successful, (2) partially successful, (3) unsuccessful.

$10 = 8 + \underline{\mathbf{18}}$	$999 = \underline{\mathbf{1000}} + 1$	$470 = 30 + \underline{\mathbf{500}}$

Figure 1 Errors at Operational Level of Understanding of the Equals Sign

were no students who made errors in all examples. All students successfully calculated the value of the expression on the left side of the equality, but they did make mistakes when calculating the expression on the right side of the equation. This shows that students can manage to solve equalities with atypical structure, i.e. those that do not fit the common framework in which the expression is on the left, and the result on the right side of the equality. Errors that were made at operational level can largely be attributed to miscalculation, both in the case of first-grade and third-grade students.

The second task for examining the development of the equals sign at operational level was formulated so that one of the elements of the expression was unknown, while the value of the expression was given. The results show that first-grade students (85.32%) were more successful than third-graders (74.76%), but their differences were not statistically significant (table 4). There were more partially successful students among the third-graders, while there was only one unsuccessful student, a first-grader.

Examples of incorrectly solved tasks show that students view the equals sign as a command to calculate (figure 1).

In the first example in figure 1, it is clear that students are trying to calculate the sum of 10 and 8, writing the sum in the blanks, despite the fact that there is an equals sign between these numbers. The situation is almost identical in the other two examples that illustrate typical errors made by third-graders. Here, we have exactly what we assumed would happen, namely, the majority of errors were made in precisely such examples, where performing the arithmetic operation would yield a 'rounded' result, therefore, students tended to neglect addition and sought to solve the cognitive conflict that arises in situations where we have atypical equalities where the expression is on the

Table 5 Understanding the Equals Sign at Basic Relational Level

Grade	(1)		(2)		(3)		Chi-square
	f	*f*%	*f*	*f*%	*f*	*f*%	
First	38	34.86	48	44.04	23	21.10	$\chi^2 = 16.28$, $df = 2$, $p = 0.0002$
Third	64	62.14	29	28.16	10	9.71	

Notes Column headings are as follows: (1) successful, (2) partially successful, (3) unsuccessful.

right, and the result on the left side of the equation. Our research confirms the results of other studies (Carpenter and Levi 2000; Filloy and Rojano 1989; McNeil et al. 2011; Harbour, Karp, and Lingo 2016) in which children often perceive equalities such as the ones above as incorrect, or make mistakes while trying to solve them.

This state of affairs is not surprising for first-grade students; however, one would expect that third-grade students would be more successful when solving tasks of this complexity, and that the only mistakes they make would involve miscalculations. Similar results were obtained by Milinković, Maričić, and Lazić (2022) in their research that investigated the performance of second-grade and fourth-grade students in tasks that required the same level of understanding of the equals sign. This research also showed students' performance at operational level, but in this case, the differences between students of different grades were statistically significant, in favour of fourth-graders.

Understanding the Equals Sign at Basic Relational Level

The second task in this research aimed to investigate students' performance in understanding the equals sign at basic relational level. The task used to examine the development of the basic relational level involved equalities in which expressions were located both on the left and the right side of the equality, except that one of the elements in the equality was omitted. Students were asked to write the missing number in empty boxes, so as to get a correct equality as the result. Only about a third of first-graders (34.86%) successfully solved this task, while 44.04% of them solved more than one example successfully, but not all, and 21.10% did not solve any example correctly (table 5). When it comes to third-grade students, over 60% of them were successful, and twice as many were partially successful. Less than 10% of the respondents were completely unsuccessful.

The differences in performance between first-grade and third-grade students at this level are statistically significant, as confirmed by the value of the

8 – 4 = **4** – 2	2 + 6 = **13** + 5	2 + 6 = **8** + 5 = **13**	10 – 8 = 1 + **2** = 3
840 – 40 = **800** – 20	**11** + 1 = 7 + 3	**6** + 1 = 7 + 3 = **10**	5 + **4** = 9 – 2 = **7**
	5 + **10** = 4 + 1	5 + **0** = 4 + 1 = **5**	8 – 4 = **4** – 2 = **2**

Figure 2 Errors at Basic Relational Level of Understanding of the Equals Sign in Examples Where the Expression Can Be on Either Side of the Equation

chi-square test ($\chi^2 = 16.28$, $df = 2$, $p < 0.01$). The test shows that third-grade students are more successful at the basic relational level of understanding of the equals sign, which is expected, because they have more experience with similar equalities. The results obtained at this level of understanding are consistent with the results of other research around the world (McAuliffe et al. 2020; Rittle-Johnson et al. 2011). Figure 2 showcases some of the typical errors made by students when solving this task.

The first two examples (figure 2), both of which feature subtraction, were particularly difficult for students of both grades, and they made numerous mistakes. Their task was to write the difference of the numbers from the first expression, which indicates that they still understand the equals sign operationally, and do not observe the expression as an independent entity, but as a prompt to perform the arithmetic operation. Operational understanding is present in the second example as well, where students first calculate the value of the expression in which both numbers are given, then add a third number, and finally, write the final result in the box, completely ignoring the equality on the left and the right side. In the third example, students solve equalities as sequences that require them to supply the missing numbers, so that the number after the equals sign is the result of the operations preceding it. As a result, they add another equals sign before the 'final result.' Similar errors are also characteristic of the research by Milinković, Maričić, and Lazić (2022), where students showed that they are reliant only on one direction of performing the operations, while completely disregarding the equivalence of expressions separated by the equals sign. Similarly, Falkner, Levi, and Carpenter (1999) found that a substantial number of first- through sixth-grade students' solutions to the equation 8 + 4 = ___ + 5 included 12, 17, or 12 and 17-solutions that are consistent with a view of the equals sign as announcing the result of an arithmetic operation.

Understanding the Equals Sign at Complex Relational Level

The third research task aimed to investigate the deeper transitional property of the equals sign in the relational sense, so equalities in which the equals sign is repeated two or more times were used for that purpose. The results

Table 6 Understanding the Equals Sign at Complex Relational Level

Grade	(1)		(2)		(3)		Chi-square
	f	*f*%	*f*	*f*%	*f*	*f*%	
First	14	12.84	54	49.54	41	37.61	$\chi^2 = 18.77$, $df = 2$, $p = 0.00008$
Third	37	35.92	47	45.63	19	18.45	

Notes Column headings are as follows: (1) successful, (2) partially successful, (3) unsuccessful.

$9 + 1 = \underline{\mathbf{10}} + 7 = \underline{\mathbf{17}} + 5$	$16 + 44 = 100 - \underline{\mathbf{60}} = \underline{\mathbf{40}} + 30$
$\underline{\mathbf{5}} + 4 + 1 = 10 - \underline{\mathbf{8}} = 2 + \underline{\mathbf{?}}$	$50 + \underline{\mathbf{50}} = 200 - 50 = \underline{\mathbf{150}} + 30$

Figure 3 Errors Related to Complex Relational Understanding of the Equals Sign

show that first-grade students are not very successful at this level of understanding (12.84%), which is not surprising considering that many of them had never encountered similar tasks before, and that their understanding of the equals sign is closely linked to arithmetic and its content, where its operational meaning is prevalent (table 6). Almost half of the respondents were partially successful, meaning that they correctly solved at least one of the given examples, so we can conclude that relational perception of the equals sign is just beginning to take shape among this student group. Third-grade students also did poorly in solving this task, since approximately every third student (35.92%) succeeded in solving this task correctly, indicating that the vast majority of students still do not fully perceive the equals sign in a relational sense.

If we compare first-grade and third-grade students, the chi-square test will show that third-grade students were significantly more successful in complex relational understanding of the equals sign. We will provide an overview of typical errors that students made in solving these tasks in particular (figure 3).

Solutions in the examples on the left (figure 3) illustrate operational understanding of the equals sign, i.e. student perception of the sign as a command 'to calculate.' In the first example, students are required to calculate the value of the first expression and write the solution in the blanks, then add the number seven to the result and write the sum in the blanks as the final result, completely ignoring the number five. We have the same situation in the second example, except that students are required to write another addend in order to get the 'correct' sum as the result of the first expression, and then write the number eight in the blanks, so that the number two would become the correct result of subtraction.

Table 7 Understanding The Equals Sign at Complex Relational Level

Grade	Successful		Unsuccessful		Chi-square
	f	*f*%	*f*	*f*%	
First	21	19.27	88	80.73	$\chi^2 = 2.818$, $df = 1$, $p = 0.093$
Third	30	29.13	73	70.87	

In the examples on the right (figure 3), we have similar typical errors, which show that some of the third-graders understand the equals sign in the exact same way as the first-graders, i.e. operationally. In the first example, students write the result of addition from the first expression in the first blank space, and the difference from the second expression in the second blank space. Similarly, in the next example, students write the number in the first blank space that, when added to the first number, yields the 'correct' result after the equals sign, ignoring the expression on the right side of the equality. Likewise, they write down the result of subtraction from the second expression, ignoring the expression on the right side again.

The second task related to the complex relational level of understanding was represented by an illustration comprising several expressions connected by the equals sign. This task proved to be most challenging for students, which is confirmed by the results shown in table 4. The results are similar to those obtained in the previous task, which only confirms that the percentage of students who have mastered relational meaning of the equals sign is very low in both grades. The success rate of the first-graders is under 20%, while the third-graders were successful in almost 30% of cases.

First-grade students achieved slightly better results compared to the previous task; however, the fact that less than a third of the respondents managed to successfully solve this task is cause for concern. It is important to note that there is no statistically significant difference in the performance of first- and third-graders, as confirmed by the value of the chi-square test (figure 4).

Analysing ways in which students solved this task, we concluded that both groups had made the same mistakes, i.e. utilized the same problem-solving strategies. Examples shown in figure 4 confirm that students neglected the equality of expressions in this task as well, concentrating on how to calculate the results instead. Consequently, they wrote the results of the expressions until they encountered an incongruity, i.e. a number that does not fit into the arithmetic sequence, after which they would start solving the equality in the opposite direction – backwards – dividing the equality into two parts. In the second example, we have the same situation with third-grade students, ex-

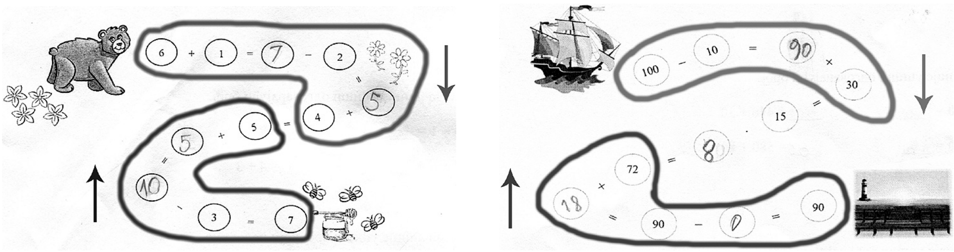


Figure 4 Errors in Understanding the Equals Sign at Complex Relational Level in the Example of Irregular Sequences

cept that the third sequence did not fit into any sequence of solutions, so we have different solutions aimed at addressing this problem. What is certain is that few students recognized the relationships of equality that exist between the expressions and wrote the correct numbers in the blanks.

All the types of typical errors that feature in our examples show that the majority of students still predominantly rely on operational understanding of the equals sign, and that they struggle to understand a sequence of expressions connected by the equals sign. Such results generally indicate that students have difficulty understanding the essential property of the equals sign and its nature as a symbol that connects equivalent entities (Seo and Ginsburg 2003). On the other hand, some studies show that by systematically practicing and providing students with opportunities to discuss complex equalities with multiple equals signs with their peers, we can influence their understanding of this sign (Lee and Pang 2020). Research also shows that the correct understanding of the equals sign can be improved by having students compare it with the 'less than' and 'greater than' signs (Hattikudur and Alibali 2010).

Conclusion

Understanding the equals sign is a key outcome of mathematics education. Considering the fact that numerous research has shown that students only possess operational understanding of the concept, i.e. understand the sign as a command to 'calculate,' while completely lacking relational understanding of the said concept, this task becomes even more urgent. Therefore, Li et al. (2008) are right when they point out that the equals sign is more than the '=' command on the calculator. The results of the conducted research on the understanding of the equals sign among first-grade and third-grade students of primary school revealed that:

- Students possess a developed understanding of the equals sign at operational level, and are more successful in solving tasks (problems) in which the expression is located on the left side of the equals sign than those in which it is located on the right;
- Only a third of the examined first-grade students possess a developed relational understanding of the equals sign, whereas almost two thirds of the third-graders understand the equals sign relationally;
- Very few among the examined first-graders possess a developed understanding of the equals sign at the complex relational level in contrast to one third of the third-graders who understand the equals sign in a relational sense.

In view of the fact that students form the concept of the equals sign from the earliest days of their formal education, the obtained results point out that students do not fully develop the meaning of this symbol in the process, and that its operational meaning is still dominant. The intensive application of the equals sign in elementary mathematics education for teaching calculus is obviously the main cause of insufficient relational understanding. This may indicate that teaching has long since turned into a routine, focused solely on calculating the value of expressions, as well as that it largely relies on expressions located on the left side of the equals sign. Students' poor performance at the complex relational level showcases the lack of essential understanding of the meaning of this concept which signifies 'sameness,' 'equivalence,' and 'transitivity.' Such results can lead to serious problems in later stages of mathematics education. Considering research findings by Rittle-Johnson et al. (2011), it appears that the equals sign develops at all four levels of understanding at the same time. This does not mean that the understanding of the equals sign exclusively pertains to any single level, or that its development at the said level is finished, which is also confirmed in this research. In contrast, Matthews et al. (2012) believe that operational and relational understanding of the equals sign develop side by side and are co-dependent. In view of this fact, we can say that there must be an obvious need for continuous practice and development of the concept of the equals sign at the individual level until it is complete at the highest relational level.

The obtained results suggest that maths teachers should form a deeper understanding of the equals sign as a symbol of equivalence between two sides of the equality. Falkner, Levi, and Carpenter point out that 'teachers should also be concerned about children's conceptions of equality as soon as symbols for representing number operations are introduced' (Falkner, Levi, and Carpenter 1999, 233). This concept must be formed gradually, over a long period of time, as confirmed by the results of this research, because third-grade students achieved significantly better performance than first-graders. The equals sign is equally important in all areas of mathematics, having a particular, but similar, meaning in each. From the aspect of algebra, the results suggest that the correct understanding of the equals sign can influence the early learning of all algebra content, i.e. that the correct interpretation of the equals sign can influence one's later learning of mathematical equivalence related to different algebraic concepts (Byrd et al. 2015). Similarly, results of the study by McNeil et al. (2012) indicate a causal relationship between practicing complex arithmetic equalities and improving one's performance in solving equations with arithmetic operations on both sides of the equality. It shows that by simply creating exercises involving arithmetic content that expresses equivalence, we help students to better understand mathematical equivalence on the whole.

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Enačaj: izziv pri učenju matematike

Avtorji v prispevku opozarjajo na matematični znak enakosti, poimenovan enačaj, kot ključni element učenja aritmetike in osnovo za učenje drugih matematičnih področij. Hkrati poudarjajo, da učenci ta znak najpogosteje razumejo operativno, primanjkuje pa jim relacijsko razumevanje le-tega. Cilj raziskave je preveriti, kako učenci prvega in tretjega razreda osnovne šole razumejo pojem enačaj glede na operacionalizacijske ravni (operacionalna, osnovna relacijska in kompleksna relacijska raven) ter če obstajajo razlike med njimi. Testiranje učencev prvega in tretjega razreda osnovne šole, teh je bilo v raziskavo vključenih 212, je dalo rezultate, ki kažejo, da pri učencih prevladuje operativno razumevanje enačaja, medtem ko je relacijsko razumevanje premalo razvito in se kot razvitejše kaže šele pri tretješolcih. Zato avtorji opozarjajo, da je treba pri učenju aritmetike več pozornosti nameniti razvoju relacijskega koncepta enačaja.

Ključne besede: aritmetika, enačaj, ekvivalentnost, matematika, pouk matematike