PhD Summer School in Discrete Mathematics

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Published by
University of Primorska Press,
Titov trg 4, 6000 Koper
Koper • 2013

Editor-in-Chief
Dr Jonatan Vinkler

Managing Editor
Alen Ježovnik

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www.hippocampus.si
Print run • 50 • Not for sale
Preface

This is a collection of lecture notes of the PhD Summer School in Discrete Mathematics, held from June 16 to June 21, 2013, by tradition at Rogla, Slovenia. The organization of this summer school came as a combined effort the Faculty of Mathematics, Natural Sciences and Information Technologies and the Andrej Marušič Institute at the University of Primorska, and the Centre for Discrete Mathematics at the Faculty of Education at the University of Ljubljana.

The Scientific Committee of the meeting consisted of Klavdija Kutnar, Aleksander Malnič, Dragan Marušič, Štefko Miklavič and Primož Šparl. The Organizing Committee of the meeting consisted of Ademir Hujdurović, Boštjan Frelih and Boštjan Kuzman.

The aim of this Summer School was to bring together senior researchers, junior researchers and PhD students working in Algebraic Graph Theory. The summer school has consisted of three minicourses given by

- Prof. Marston Conder, University of Auckland, New Zealand,
- Prof. Edward T. Dobson, Mississippi State University, USA & UP, Slovenia, and
- Prof. Tatsuro Ito, Kanazawa University, Japan.
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Chapter 1

Graph Symmetries

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Summary

• Introduction to symmetries of graphs
• Vertex-transitive and arc-transitive graphs
• $s$-arc-transitivity (including theorems of Tutte and Weiss)
• Proof of Tutte's theorem on symmetric cubic graphs
• Use of amalgams and covers to analyse and construct examples
• Some recent developments
1.1 Introduction to Symmetries of Graphs

Generally, an object is said to have symmetry if it can be transformed in way that leaves it looking the same as it did originally.

**Automorphisms:** An automorphism (or symmetry) of a simple graph $X = (V, E)$ is a permutation of the vertices of $X$ which preserves the relation of adjacency; that is, a bijection $\pi : V \rightarrow V$ such that $\{v^\pi, w^\pi\} \in E$ if and only if $\{v, w\} \in E$.

Under composition, the automorphisms form a group, called the automorphism group (or symmetry group) of $X$, and this is denoted by $\text{Aut}(X)$, or $\text{Aut} X$.

**Examples**

(a) Complete graphs and null graphs: $\text{Aut} K_n \cong \text{Aut} N_n \cong S_n$ for all $n$.
(b) Simple cycles: $\text{Aut} C_n \cong D_n$ (dihedral group of order $2n$) for all $n \geq 3$.
(c) Simple paths: $\text{Aut} P_n \cong S_2$ for all $n \geq 3$.
(d) Complete bipartite graphs: $\text{Aut} K_{m,n} \cong S_m \times S_n$ when $m \neq n$, while $\text{Aut} K_{n,n} \cong S_n \times S_2 \cong (S_n \times S_n) \rtimes S_2$ (when $m = n$).
(e) Star graphs – see above: $\text{Aut} K_{1,n} \cong S_n$ for all $n > 1$.
(f) Wheel graphs (cycle $C_{n-1}$ plus $n$th vertex joined to all): $\text{Aut} W_n \cong D_{n-1}$ for all $n \geq 5$.
(g) Petersen graph: $\text{Aut} P \cong S_5$.

**Exercise 1:** How many automorphisms has the underlying graph (1-skeleton) of each of the five Platonic solids: the regular tetrahedron, cube, octahedron, dodecahedron and icosahedron?

**Exercise 2:** Find a simple graph on 6 vertices that has exactly one automorphism.

**Exercise 3:** Find a simple graph that has exactly three automorphisms. What is the smallest such graph?

**Exercise 4:** For large $n$, do ‘most’ graphs of order $n$ have a large automorphism group? or just the identity automorphism?

One amazing fact about graphs and groups is Frucht’s theorem: in 1939, Robert(o) Frucht proved that given any finite group $G$, there exist infinitely many connected graphs $X$ such that $\text{Aut} X$ is isomorphic to $G$. And then later, in 1949, he proved that $X$ may be chosen to be 3-valent. There are several variants and generalisations of this, such as regular representations for graphs and digraphs (GRRs and DRRs).

The simple graphs of order $n$ with the largest number of automorphisms are the null graph $N_n$ and the complete graph $K_n$, each with automorphism group $S_n$ (the symmetric group on $n$ symbols). For non-null, incomplete graphs of bounded valency (vertex degree), the situation is more interesting.

In this course of lectures, we will devote quite a lot of attention to the case of regular graphs of valency 3, which are often called cubic graphs.

**Exercise 5:** For each $n \in \{4, 6, 8, 10, 12, 14, 16\}$, which 3-valent connected graph on $n$ vertices has the largest number of automorphisms?

For fixed valency, some of the graphs with the largest number of automorphisms do not
look particularly nice, or do not have other good properties (e.g., strength/stability, or suitability for broadcast networks). The ‘best’ graphs possess special kinds of symmetry.

**Transitivity:** A graph $X = (V, E)$ is said to be

- **vertex-transitive** if $\text{Aut}X$ has a single orbit on the vertex-set $V$;
- **edge-transitive** if $\text{Aut}X$ has a single orbit on the edge-set $E$;
- **arc-transitive** (or symmetric) if $\text{Aut} X$ has a single orbit on the arc-set (that is, the set $A = \{(v, w) | \{v, w\} \in E\}$ of all ordered pairs of adjacent vertices);
- **distance-transitive** if $\text{Aut} X$ has a single orbit on each of the sets $\{(v, w) | d(v, w) = k\}$ for $k = 0, 1, 2, \ldots$;
- **semi-symmetric** if $X$ is edge-transitive but not vertex-transitive;
- **half-arc-transitive** if $X$ is vertex-transitive and edge-transitive but not arc-transitive.

**Examples**

(a) Complete graphs: $K_n$ is vertex-transitive, edge-transitive, arc-transitive and distance-transitive.

(b) Simple cycles: $C_n$ is vertex-transitive, edge-transitive, arc-transitive and distance-transitive.

(c) Complete bipartite graphs: $K_{m,n}$ is edge-transitive, but is vertex-transitive (and arc-transitive and distance-transitive) only when $m = n$.

(d) Wheel graphs: $W_n$ is neither vertex-transitive nor edge-transitive (for $n \geq 5$).

(e) The Petersen graph is vertex-transitive, edge-transitive, arc-transitive and distance-transitive.

Note that every **vertex-transitive graph is regular** (in the sense of having all vertices of the same degree/valency), since for any two vertices $v$ and $w$, there is an automorphism $\theta$ taking $v$ to $w$, and then $\theta$ takes the edges incident with $v$ to the edges incident with $w$.

On the other hand, not every edge-transitive graph is regular: counter-examples include all $K_{m,n}$ for $m \neq n$. But there exist graphs that are edge-transitive and regular but not vertex-transitive. One example is the smallest semi-symmetric cubic graph, called the **Gray graph** (discovered by Gray and re-discovered later by Bouwer), on 54 vertices. The smallest semi-symmetric regular graph is the **Folkman graph**, which is 4-valent on 20 vertices.

Also note that every arc-transitive connected graph without isolated vertices is both vertex-transitive and edge-transitive, but the converse does not hold. Counter-examples are half-arc-transitive. The smallest half-arc-transitive graph is the **Holt graph**, which is a 4-valent graph on 27 vertices. There are infinitely many larger examples.

**Exercise 6:** Let $X$ be a $k$-valent graph, where $k$ is odd (say $k = 3$). Show that if $X$ is both vertex-transitive and edge-transitive, then also $X$ is arc-transitive. [Harder question: does the same thing always happen when $k$ is even?]

**Exercise 7:** Prove that every semi-symmetric graph is bipartite.

**Exercise 8:** Every distance-transitive graph is arc-transitive. Can you find an arc-transitive
graph that is not distance-transitive?

**s-arcs:** An **s-arc** in a graph $X = (V, E)$ is a sequence $(v_0, v_1, \ldots, v_s)$ of vertices of $X$ in which any two consecutive vertices are adjacent and any three consecutive vertices are distinct, that is, $\{v_{i-1}, v_i\} \in E$ for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. The graph $X = (V, E)$ is called **s-arc-transitive** if $\text{Aut} X$ is transitive on the set of all $s$-arcs in $X$.

**Examples**

(a) Simple cycles: $C_n$ is $s$-arc-transitive for all $s \geq 0$, whenever $n \geq 3$.

(b) Complete graphs: $K_n$ is 2-arc-transitive, but not 3-arc-transitive, for all $n \geq 3$.

(c) The complete bipartite graph $K_{n,n}$ is 3-arc- but not 4-arc-transitive, for all $n \geq 2$.

(d) The Petersen graph is 3-arc-transitive, but not 4-arc-transitive.

(e) The Heawood graph (the incidence graph of the projective plane of order 2) is 4-arc-transitive, but not 5-arc-transitive.

**Exercise 9:** For each of the five Platonic solids, what is the largest value of $s$ such that the underlying graph (1-skeleton) is $s$-arc-transitive?

**Exercise 10:** Let $X$ be an $s$-arc-transitive $d$-valent connected simple graph. Find a lower bound on the order of the stabiliser in $\text{Aut} X$ of a vertex $v \in V(X)$, in terms of $s$ and $d$.

**Sharp transitivity (‘regularity’):** A graph $X = (V, E)$ is said to be

- **vertex-regular** if the action of $\text{Aut} X$ on the vertex-set $V$ is regular (that is, for every ordered pair $(v, w)$ of vertices, there is a unique automorphism taking $v$ to $w$);
- **edge-regular** if the action of $\text{Aut} X$ on the edge-set $E$ is regular;
- **arc-regular** if the action of $\text{Aut} X$ on the arc-set $A$ is regular;
- **$s$-arc-regular** if the action of $\text{Aut} X$ on the set of $s$-arcs of $X$ is regular.

The same terminology applies to actions of a subgroup of $\text{Aut} X$ on $X$. For example, **Cayley graphs** (which will be encountered soon) are precisely the graphs that admit a vertex-regular group of automorphisms ... and possibly other automorphisms as well.

Important note: The term ‘distance regular’ means something quite different – a graph $X$ is called **distance regular** if for all $j$ and $k$, it has the property that for any two vertices $v$ and $w$ at distance $j$ from each other, the number of vertices adjacent to $w$ and at distance $k$ from $v$ is a constant (depending only on $j$ and $k$, and not on $v$ and $w$).

### 1.2 Vertex-transitive and Arc-transitive Graphs

Let $X$ be a vertex-transitive graph, with automorphism group $G$, and let $H$ be the stabiliser of any vertex $v$, that is, the subgroup $H = G_v = \{g \in G | v^g = v\}$. Let us also assume that $X$ is not null, and hence that every vertex of $X$ has the same positive valency.

Since $G$ is transitive on $V = V(X)$, we may label the vertices with the right cosets of $H$ in $G$ such that each automorphism $g \in G$ takes the vertex labelled $H$ to the vertex labelled $Hg$ — that is, the action of $G$ on $V(X)$ is given by right multiplication on the coset space $(G : H) = \{Hg : g \in G\}$.
Next, define \( D = \{ g \in G \mid v^g \text{ is adjacent to } v \} = \{ g \in G \mid \{ v, v^g \} \in E(X) \} \). Then:

**Lemma 2.1:**

(a) \( D \) is a union of double cosets \( HaH \) of \( H \) in \( G \)

(b) \( D \) is closed under taking inverses

(c) \( v^x \) is adjacent to \( v^y \) in \( X \) if and only if \( xy^{-1} \in D \)

(d) The valency of \( X \) is the number of right cosets \( Hg \) contained in \( D \)

(e) \( X \) is connected if and only if \( D \) generates \( G \).

**Proof.**

(a) If \( a \in D \) then for all \( h, h' \in H \) we have \( v^{aha'} = v^{aha} = (v^a)h' \), which is the image of a neighbour of \( v \) under an automorphism fixing \( v \), and hence a neighbour of \( v \), so \( hah' \in D \). Thus \( HaH \subseteq D \) whenever \( a \in D \), and so \( D \) is the union of all such double cosets of \( H \).

(b) If \( a \in D \) then \( \{a^v, a^w\} \in E(X) \), and hence \( \{v, v^{a^{-1}}\} = \{a^v, v\}^{a^{-1}} \in E(X) \), so \( a^{-1} \in D \).

(c) \( \{v^x, v^y\} \in E(X) \iff \{v, v^{xy^{-1}}\} \in E(X) \iff \{v, v^{xy^{-1}}\} \in E(X) \iff xy^{-1} \in D \).

(d) By vertex-transitivity, the valency of \( X \) is the number of neighbours of \( v \). These neighbours are all of the form \( v^{a^i} \) for \( a \in D \), and if \( v^a = v^{a'} \) for \( a, a' \in D \) then \( v^{a'a^{-1}} = v \) so \( a'a^{-1} \in Gv = H \), or equivalently, \( a' \in Ha \), and conversely, if \( a' = ha \) where \( h \in H \), then \( v^{a'} = v^{ha} = v^a \). Hence this valency equals the number of right cosets of \( H \) contained in \( D \).

(e) Neighbours of \( v \) are of the form \( v^a \) where \( a \in D \), and their neighbours are of the form \( v^{a'a} \) where \( a, a' \in D \). By induction, vertices at distance at most \( k \) from \( v \) are of the form \( v^{a_1a_2 \cdots a_k} \) where \( a_i \in D \) for \( 1 \leq i \leq k \). It follows that \( X \) is connected if and only if every vertex can be written in this form (for some \( k \)), or equivalently, if and only if every element of \( G \) can be written as a product of elements of \( D \). \( \square \)

**Lemma 2.2:** \( X \) is arc-transitive if and only if the stabiliser \( H \) of a vertex \( v \) of \( X \) is transitive on the neighbours of \( v \).

**Proof.** If \( X \) is arc-transitive, then for any two neighbours \( w \) and \( w' \) of \( v \), there exists an automorphism \( g \in G \) taking \( (v, w) \) to \( (v, w') \). Any such \( g \) lies in \( G_v \), and takes \( w \) to \( w' \), and it follows that \( G_v \) is transitive on the set \( X(v) \) of all neighbours of \( v \). Conversely, suppose that \( H = G_v \) is transitive on \( X(v) \). Then for any arcs \( (v, w) \) and \( (v', w') \), some \( g \in G \) takes \( v \) to \( v' \), and if the pre-image of \( w' \) under \( g \) is \( w'' \), then also some \( h \in G_v \) takes \( w \) to \( w'' \). From these it follows that \( (v, w)^h = (v, w'^{g}) = (v', w') \). Thus \( X \) is arc-transitive. \( \square \)

**Lemma 2.3:** \( X \) is arc-transitive if and only if \( D = HaH \) for some \( a \in G \setminus H \), indeed if and only if \( D = HaH \) for some \( a \in G \) such that \( a \notin H \) but \( a^2 \in H \).

**Proof.** By Lemma 2.1, we know that \( X \) is arc-transitive if and only if \( H = G_v \) is transitive on the neighbours of \( v \), which occurs if and only if every neighbour of \( v \) is of the form \( w^h \) for some \( w \in X(v) \) and some \( h \in G_v = H \). By taking \( v^a = w \), we find the equivalent condition that \( D = HaH \) for some \( a \in G \setminus H \).

For the second part, note that \( a^{-1} \in D = HaH \), so \( a^{-1} = hah' \) for some \( h, h' \in H \). But then \( aha = (h')^{-1} \), so \( (ah)^2 = (h')^{-1}h \in H \), and also \( H(ah)H = HaH = HaH = D \), so we
can replace \( a \) by \( ah \) and then assume that \( a^2 \in H \) (and still \( a \not\in H \)).

**Constructions:** The observations in the preceding lemmas can be turned around to produce constructions for vertex-transitive and arc-transitive graphs, as follows.

Let \( G \) be any group, \( H \) any subgroup of \( G \), and \( D \) any union of double cosets of \( G \) such that \( H \cap D = \emptyset \), and \( D \) is closed under taking inverses. [Note: there is also a construction for vertex-transitive digraphs that does not assume \( D \) is inverse-closed.]

Now define a graph \( X = X(G, H, D) \) by taking \( V = V(X) \) to be the right coset space \( (G : H) = \{Hg : g \in G\} \), and \( E = E(X) \) to be the set of all pairs of the form \( \{Hx, Hax\} \) where \( a \in D \) and \( x \in G \). [This construction is due to Sabidussi (1964)]

The adjacency relation is symmetric, since \( Hx = Ha(a^{-1}x) \), and so this is an undirected simple graph. Also right multiplication gives an action of \( G \) on \( X \), with \( g \in G \) taking a vertex \( Hx \) to the vertex \( Hxg \), and an edge \( \{Hx, Hax\} \) to the edge \( \{Hxg, Haxg\} \). This action is transitive on vertices, since \( g \) takes \( H \) to \( Hg \) for any \( g \in G \). The stabiliser of the vertex \( H \) is \( \{g \in G \mid Hg = H\} \), which is the subgroup \( H \) itself (since \( Hg = H \) if and only if \( g \in H \)). Valency and connectedness are as in Lemma 2.1.

Note, however, that the action of \( G \) on \( X(G, H, D) \) need not be faithful: the kernel \( K \) of this action is the core of \( H \) (the intersection of all conjugates \( g^{-1}Hg \) of \( H \)) in \( G \). Similarly, the group \( G/K \) induced on \( X(G, H, D) \) need not be the full automorphism group \( \text{Aut} X \); it is often possible that the graph admits additional automorphisms.

**Cayley graphs:** Given a group \( G \) and a set \( D \) of elements of \( G \), the **Cayley graph** \( \text{Cay}(G, D) \) is the graph with vertex-set \( G \), and edge set \( \{\{x, ax\} : x \in G, a \in D\} \). Note that this is a special case of the above, with \( H = \{1\} \).

In particular, \( \text{Cay}(G, D) \) is vertex-transitive, and the group \( G \) acts faithfully and regularly on the vertex-set, but is not necessarily the full automorphism group. For example, a **circulant** (which is a Cayley graph for a cyclic group) can often have more than just simple rotations. Similarly, the \( n \)-dimensional hypercube \( Q_n \) is the Cayley graph \( \text{Cay}(\mathbb{Z}_2^n, B) \) where \( B \) is the standard basis (of elementary vectors) for \( \mathbb{Z}_2^n \), but \( \text{Aut} Q_n \cong \mathbb{Z}_2 \wr S_n \cong \mathbb{Z}_2^n \rtimes S_n \).

1.3 **s-arc-transitivity (and Theorems of Tutte and Weiss)**

As defined earlier, an **s-arc** in a graph \( X \) is a sequence \((v_0, v_1, \ldots, v_s)\) of \( s + 1 \) vertices of \( X \) in which any 2 consecutive vertices are adjacent and any 3 consecutive vertices are distinct. The graph \( X \) is called **s-arc-transitive** if \( \text{Aut} X \) is transitive on the \( s \)-arcs in \( X \).

**Lemma 3.1:** Let \( X \) be a vertex-transitive graph of valency \( k > 2 \), and let \( G = \text{Aut} X \). Then \( X \) is 2-arc-transitive if and only if the stabiliser \( G_v \) of a vertex \( v \) is 2-transitive on the \( k \) neighbours of \( v \).

**Proof.** If \( X \) is 2-arc-transitive, then for any two ordered pairs \((u_1, w_1)\) and \((u_2, w_2)\) of neighbours of \( v \), some automorphism \( g \in G \) takes the 2-arc \((u_1, v, w_1)\) to the 2-arc \((u_2, v, w_2),\)
in which case $g$ fixes $v$ and $g$ takes $(u_1, w_1)$ to $(u_2, w_2)$; hence $G_v$ is 2-transitive on the neighbourhood $X(v)$. Conversely, suppose $G_v$ is 2-transitive on $X(v)$, and let $(u, v, w)$ and $(u', v', w')$ be any two 2-arcs in $X$. Then by vertex-transitivity, some $g \in G$ takes $v$ to $v'$, and then if $g$ takes $u'$ to $u''$ and $w'$ to $w''$, say, then some $h \in G_v$ takes $(u, v, w)$ to $(u'', v', w')$, in which case $(u, v, w)h^{g^{-1}} = (u'', v, w'')g^{-1} = (u', v', w')$; hence $X$ is 2-arc-transitive. \hfill \Box

**Exercise 11**: For a vertex-transitive graph $X$ of valency 3, what are the possibilities for the permutation group induced on $X(v)$ by the stabiliser $G_v$ in $G = \text{Aut} X$ of a vertex $v$? Which of these correspond to arc-transitive actions?

**Exercise 12**: For an arc-transitive graph $X$ of valency 4, what are the possibilities for the permutation group induced on $X(v)$ by the stabiliser $G_v$ in $G = \text{Aut} X$ of a vertex $v$?

**Lemma 3.2**: Let $X$ be a vertex-transitive graph of valency $k > 2$, and let $G = \text{Aut} X$. Then $X$ is $(s + 1)$-arc-transitive if and only if $X$ is $s$-arc-transitive and the stabiliser $G_\sigma$ of an $s$-arc $\sigma = (v_0, v_1, \ldots, v_s)$ is transitive on $X(v_s) \setminus \{v_{s-1}\}$ (the set of $k - 1$ neighbours of $v_s$ other than $v_{s-1}$).

**Proof**. If $X$ is $(s + 1)$-arc-transitive, then for any $s$-arc $\sigma = (v_0, v_1, \ldots, v_s)$ and any vertices $w$ and $w'$ in $X(v_s) \setminus \{v_{s-1}\}$, some automorphism $g \in G$ takes the $(s + 1)$-arc $(v_0, v_1, \ldots, v_s, w)$ to the $(s + 1)$-arc $(v_0, v_1, \ldots, v_s, w')$, in which case $g$ fixes $\sigma$, and $g$ takes $w$ to $w'$; hence $G_\sigma$ is transitive on $X(v_s) \setminus \{v_{s-1}\}$. Conversely, suppose $G_\sigma$ is transitive on $X(v_s) \setminus \{v_{s-1}\}$, and let $(v_0, v_1, \ldots, v_s, v_{s+1})$ and $(w_0, w_1, \ldots, w_s, w_{s+1})$ be any two $(s + 1)$-arcs in $X$. Then by $s$-arc-transitivity, some $g \in G$ takes $(v_0, v_1, \ldots, v_s)$ to $(w_0, w_1, \ldots, w_s)$, and then if $g$ takes $w_{s+1}$ to $w'$, say, then some $h \in G_\sigma$ takes $v_{s+1}$ to $w'$, in which case

$$(v_0, v_1, \ldots, v_s, v_{s+1})h^{g^{-1}} = (v_0, v_1, \ldots, v_s, w')g^{-1} = (w_0, w_1, \ldots, w_s, w_{s+1});$$

hence $X$ is $(s + 1)$-arc-transitive. \hfill \Box

The simple cycle $C_n$ (which has valency 2) is $s$-arc-transitive for all $s \geq 0$, as is the union of more than one copy of $C_n$. This case is somewhat exceptional. For $k > 2$, there is an upper bound on values of $s$ for which there exists a finite $s$-arc-transitive graph of valency $k$, as shown by the theorems of Tutte and Weiss below.

The first theorem, due to W.T. Tutte, is for valency 3, and will be proved in Section 4. On the other hand, the second theorem, due to Richard Weiss, is for arbitrary valency $k \geq 3$, but its proof is much more difficult, and is beyond the scope of this course.

**Theorem 3.3** [Tutte, 1959]: Let $X$ be a finite connected arc-transitive graph of valency 3. Then $X$ is $s$-arc-regular (and so $|\text{Aut} X| = 3 \cdot 2^{s-1} \cdot |V(X)|$) for some $s \leq 5$. Hence in particular, there are no finite 6-arc-transitive cubic graphs.

The upper bound on $s$ in Tutte’s theorem is sharp; in fact, it is attained by infinitely many graphs, although these graphs are somewhat rare. The smallest example is given below.

**Tutte’s 8-cage**: This is the smallest 3-valent graph of girth 8, and has 30 vertices. It can be constructed in many different ways. One way is as follows:

In the symmetric group $S_6$, there are $\binom{6}{2} = 15$ transpositions (2-cycles), and $5 \cdot 3 \cdot 1 = 15$ triple transpositions (sometimes called synthemes). Define a graph $T$ by taking these 30
permutations as the vertices, and joining each triple transposition \((a, b)(c, d)(e, f)\) by an edge to each of its three transpositions \((a, b), (c, d)\) and \((d, e)\).

The resulting graph \(T\) is Tutte's 8-cage. It is 3-valent, bipartite and connected, and the group \(S_6\) induces a group of automorphisms of \(T\) by conjugation of the elements.

**Exercise 13:** Write down the form of a typical 5-arc \((v_0, v_1, \ldots, v_5)\) in Tutte's cage \(T\) with initial vertex \(v_0\) being a transposition \((a, b)\). Use this to prove that (a) the group \(S_6\) is transitive on all such 5-arcs, and (b) \(T\) is not 6-arc-transitive.

**Exercise 14:** Prove that the girth (the length of the smallest cycle) of Tutte's 8-cage is 8.

Now the group \(S_6\) is somewhat special among symmetric groups in that \(\text{Aut}S_6\) is twice as large as \(S_6\). In fact, \(S_6\) admits an outer automorphism that interchanges the 15 transpositions with the 15 triple transpositions, and interchanges the 40 3-cycles with the 40 double 3-cycles \((a, b, c)(d, e, f)\). Any such outer automorphism reverses a 5-arc in Tutte's 8-cage, and it follows that Tutte's 8-cage is 5-arc-transitive.

Note that Tutte's theorem actually puts a bound on the order of the stabiliser of a vertex (in the automorphism group of a finite symmetric 3-valent graph). The same thing does not hold for 4-valent symmetric graphs, as shown by the following.

**Necklace/wreath graphs:** Take a simple cycle \(C_n\), where \(n \geq 3\), with vertices labelled \(0, 1, 2, \ldots, n - 1\) in cyclic order, and then replace every vertex \(j\) by a pair of vertices \(u_j\) and \(v_j\), and join every such \(u_j\) and every such \(v_j\) by an edge to each of the four vertices \(u_{j-1}, v_{j-1}, u_{j+1}\) and \(v_{j+1}\), with addition and subtraction of subscripts taken modulo \(n\). The resulting 4-valent graph (called a 'necklace' or 'wreath' graph) has \(2n\) vertices, and is arc-transitive, with automorphism group isomorphic to the wreath product \(S_2 \wr D_n \cong (S_2)^n \rtimes D_n\). In particular, the stabiliser of any vertex has order \(2^n\), which is unbounded.

**Exercise 15:** What is the largest value of \(s\) for which the above graph (on \(2n\) vertices) is \(s\)-arc-transitive?

It is also worth noting here that vertex-stabilisers are bounded for the automorphism groups of maps. A **map** is an embedding of a connected graph or multigraph on a surface, dividing the surface into simply-connected regions, called the faces of the map. By definition, an automorphism of a map \(M\) preserves incidence between vertices, edges and faces of \(M\), and it follows that if a vertex \(v\) has degree \(k\), then the stabiliser of \(v\) in \(\text{Aut}M\) is a subgroup of the dihedral group \(D_k\). The most highly symmetric maps are called **rotary**, or regular.

**Theorem 3.4** [Weiss, 1981]: Let \(X\) be a finite connected \(s\)-arc-transitive graph of valency \(k \geq 3\). Then \(s \leq 7\), and if \(s = 7\) then \(k = 3^t + 1\) for some \(t\). Hence in particular, there are no finite 8-arc-transitive graphs of valency \(k\) whenever \(k > 2\).

As with Tutte's theorem, the upper bound on \(s\) in Weiss's theorem is sharp. In fact, for every \(t > 0\), the incidence graph of a generalised hexagon over \(GF(3^t)\) is a 7-arc-transitive graph of valency \(3^t + 1\).

The proof of Weiss's theorem uses the fact that if \(X\) is \(s\)-arc-transitive for some \(s \geq 2\), then \(X\) is 2-arc-transitive (by Lemma 3.2), and so the stabiliser in \(G = \text{Aut}X\) of a vertex
\( v \) of \( X \) is 2-transitive on the neighbourhood \( X(v) \) of \( v \) (by Lemma 3.1). It then uses the classification of finite 2-transitive groups, obtained by Peter Cameron in 1981 using the classification of the finite simple groups (CFSG).

Finally in this section, we give something that is useful in proving Tutte’s theorem (and in other contexts as well):

**Lemma 3.5** (The ‘even distance’ lemma): For any connected arc-transitive graph \( X \), let \( G^+ = \langle G_v, G_w \rangle \) be the subgroup of \( G = \text{Aut} \, X \) generated by the stabilisers \( G_v \) and \( G_w \) of any two adjacent vertices \( v \) and \( w \). Then

(a) the orbit of \( v \) under \( G^+ \) contains all vertices at even distance from \( v \),
(b) \( G^+ \) contains the stabiliser of every vertex of \( X \),
(c) \( G^+ \) has index 1 or 2 in \( G = \text{Aut} \, X \), and
(d) \( |G : G^+| = 2 \) if and only if \( X \) is bipartite.

**Proof.** Let \( \Omega \) and \( \mathcal{U} \) be the orbits of \( v \) and \( w \) under \( G^+ \). Then \( \mathcal{U} \) contains \( w^{G_v} \), so contains all neighbours of \( v \). Similarly, \( \Omega \) contains all neighbours of \( w \), so contains all vertices at distance 2 from \( v \). Also \( G^+ \) contains their stabilisers; for example, if \( h \in G^+ \) takes \( v \) to \( z \), then \( G^+ \) contains \( h^{-1}G_v h = G_z \). Parts (a) and (b) now follow from these observations, by induction. By the same token, the orbit \( \mathcal{U} = w^{G^+} \) contains all vertices at even distance from \( v \). Hence in particular, every vertex of \( X \) lies in \( \Omega \cup \mathcal{U} \). Also by the orbit-stabiliser theorem, \( |G_v| |\Omega| = |G^+| = |G_w| |\mathcal{U}| \), and then since \( |G_v| = |G_w| \), this implies \( |\Omega| = |\mathcal{U}| \), and it follows that \( |\Omega| = |\mathcal{U}| = |V(X)| / 2 \) or \( |V(X)| / 2 \). In the latter case, \( \Omega \) and \( \mathcal{U} \) are disjoint, which happens if and only if \( X \) is bipartite (with parts \( \Omega \) and \( \mathcal{U} \)), and then also \( |G| = |G_v| |V(X)| = 2|G_v| |\Omega| = 2|G^+| \), so \( G^+ \) has index 2 in \( G \). On the other hand, in the former case, \( \Omega = \mathcal{U} = V(X) \) and \( |G| = |G_v| |V(X)| = |G_v| |\Omega| = |G^+| \), and then \( G^+ = G \). This proves parts (c) and (d). \( \square \)

### 1.4 Proof of Tutte’s Theorem on Symmetric Cubic Graphs

**Theorem** [Tutte, 1959]: Let \( X \) be a finite connected arc-transitive graph of valency 3. Then \( X \) is \( s \)-arc-regular (and so \( |\text{Aut} \, X| = 3 \cdot 2^{s-1} \cdot |V(X)| \)) for some \( s \leq 5 \). Hence in particular, there are no finite 6-arc-transitive cubic graphs.

We will prove this in several stages, using only elementary theory of groups and graphs.

First, we let \( s \) be the largest positive integer \( t \) for which the graph \( X \) is \( t \)-arc-transitive, and let \( G = \text{Aut} \, X \).

Then we let \( \sigma = (v_0, v_1, \ldots, v_s) \) be any \( s \)-arc in \( X \), and consider the stabilisers in \( G \) of the 0-arc \( (v_0) \), the 1-arc \( (v_0, v_1) \), the 2-arc \( (v_0, v_1, v_2) \), and so on.

We use properties of these to show that \( X \) is \( s \)-arc-regular, and then by considering the smallest \( k \) for which the stabiliser in \( G \) of the \( k \)-arc \( (v_0, v_1, \ldots, v_k) \) is abelian, we prove that \( s \leq 5 \).

**Lemma 4.1**: \( X \) is \( s \)-arc-regular.

**Proof.** We have assumed \( X \) is \( s \)-arc-transitive, so all we have to do is show that the sta-
bilsifer of an s-arc is trivial. So assume the contrary. Then every s-arc $\sigma$ is preserved by some non-trivial automorphism $f$, and by conjugating by a 'shunt' if necessary, we can choose $f = (v_0, v_1, \ldots, v_s)$ such that $f$ moves one of the neighbours of $v_s$, say $w$. Then since $f$ fixes $v_s$ and its neighbour $v_{s-1}$, it must inter-change $w$ with the third neighbour $w'$ of $v_s$. It follows that the stabiliser of the s-arc $\sigma = (v_0, v_1, \ldots, v_s)$ is transitive on the set of two $(s + 1)$-arcs extending $\sigma$, namely $(v_0, v_1, \ldots, v_s, w)$ and $(v_0, v_1, \ldots, v_s, w')$. Hence $X$ is $(s + 1)$-arc-transitive, contradiction.

\[\square\]

**Stabilisers**

Let $\sigma = (v_0, v_1, \ldots, v_s)$ be any s-arc of $X$, and let $G = \text{Aut} X$, and now define

\[
\begin{align*}
H_s &= G^{(0)} = G_{\{v_0\}} \\
H_{s-1} &= G^{(1)} = G_{\{v_0, v_1\}} \\
& \vdots \quad \vdots \\
H_{s-k} &= G^{(k)} = G_{\{v_0, v_1, \ldots, v_k\}} \\
& \vdots \quad \vdots \\
H_0 &= G^{(s)} = G_{\{v_0, v_1, \ldots, v_{s-1}, v_s\}} = \{1\}.
\end{align*}
\]

Then working backwards, we find that $|H_j| = |G^{(s-j)}| = 2^j$ for $0 \leq j < s$, while also $|H_s| = |G^{(0)}| = 3 \cdot 2^{s-1}$ and $|G| = 3 \cdot 2^{s-1} \cdot |V(X)|$.

**Particular automorphisms**

As before, let $w$ and $w'$ be the other two neighbours of $v_s$. Also let $h$ and $h'$ be the two automorphisms that take $\sigma = (v_0, v_1, \ldots, v_{s-1}, v_s)$ to $(v_1, v_2, \ldots, v_s, w)$ and $(v_1, v_2, \ldots, v_s, w')$ respectively, and define $x_0 = h'h^{-1}$ and $x_j = h^j x_0 h^{-j}$ for $1 \leq j \leq s$. Note that $x_0 = h'h^{-1}$ preserves $(v_0, v_1, \ldots, v_{s-1})$ and is non-trivial, so $x_0$ must swap $v_s$ with the third neighbour of $v_{s-1}$; hence $x_0$ has order 2. It follows that every $x_j$ has order 2.

Moreover, $x_{j-1}$ preserves $(v_0, v_1, \ldots, v_{s-j})$ and swaps $v_{s-j+1}$ with the third neighbour of $v_{s-j}$, for $1 \leq j < s$. Hence in particular, $x_{j-1} \in H_j \setminus H_{j-1}$. Then since $|H_j| = 2|H_{j-1}|$, we find that $H_j$ is generated by $\{x_0, x_1, \ldots, x_{j-1}\}$ for $1 \leq j < s$. Similarly, $x_{s-1}$ fixes $v_0$ but moves $v_1$, so $x_{s-1} \in H_s \setminus H_{s-1}$. Then since $|H_s : H_{s-1}| = 3$ (a prime), $H_{s-1}$ is a maximal subgroup of $H_s$, so $H_j$ is generated by $\{x_0, x_1, \ldots, x_{s-1}\}$.

Next, consider the subgroup $G^*$ generated by $\{x_0, x_1, \ldots, x_s\}$. This contains $H_s = G_{v_0}$ and $(x_1, x_{s-1}, x_s) = G_u$ where $u^h = v_0$, and so by the 'even distance' lemma, $G^*$ is a subgroup of index 1 or 2 in $G$ (the one we called $G^+$ earlier). In particular, $|G^*| = 3 \cdot 2^{s-1} \cdot |V(X)|$ or half of that. Finally, since $h$ moves $v_0$ to $v_1$ (which is at distance 1 from $v_0$), we find that $G = \langle h, G^* \rangle = \langle h, x_0, x_1, \ldots, x_s \rangle = \langle h, x_0 \rangle$.

We can summarise this in the following lemma.

**Lemma 4.2:**

(a) $H_j = \langle x_0, x_1, \ldots, x_{j-1} \rangle$ for $1 \leq j \leq s$,

(b) $G^+ = \langle x_0, x_1, \ldots, x_s \rangle$, and

(c) $G = \langle h, x_0, x_1, \ldots, x_s \rangle = \langle h, x_0 \rangle$.

Note that $H_1 = \langle x_0 \rangle$ and $H_2 = \langle x_0, x_1 \rangle$ are abelian, with orders 2 and 4 respectively.
Define $\lambda$ to be the largest value of $j$ for which $H_j$ is abelian. We will show that $\frac{2}{3}(s - 1) \leq \lambda < \frac{1}{2}(s + 2)$ whenever $s \geq 4$, and hence that $s \leq 5$ or $s = 7$, and then we will eliminate the possibility that $s = 7$.

**Lemma 4.3:** If $s \geq 4$, then $2 \leq \lambda < \frac{1}{2}(s + 2)$.

**Proof.** Assume the contrary. We know that $\lambda \geq 2$, so the assumption implies $2\lambda \geq s + 2$, and hence $\lambda - 1 \geq s - \lambda + 1$. Now $H_\lambda = \langle x_0, x_1, \ldots, x_{\lambda - 1} \rangle$ is abelian, and therefore so is its conjugate $h^{s - \lambda + 1} H_\lambda h^{-(s - \lambda + 1)} = \langle x_{s - \lambda + 1}, x_{s - \lambda + 2}, \ldots, x_s \rangle$. Then since $\lambda - 1 \geq s - \lambda + 1$, both of these contain $x_{\lambda - 1}$, and also together they generate $\langle x_0, x_1, \ldots, x_s \rangle = G^+$. It follows that $x_{\lambda - 1}$ commutes with every element of $G^+$. In particular, $x_{\lambda - 1}$ commutes with $h^2$ (which lies in $G^+$ since $|G : G^+| \leq 2$). But that implies $x_{s - 1} = h^2 x_{\lambda - 1} h^{-2} = x_{s + 1}$, and then conjugating by $h^{s - 1}$ gives $x_0 = x_2$, contradiction.

**Lemma 4.4:** The centre of $H_j = \langle x_0, x_1, \ldots, x_{j - 1} \rangle$ is generated by $\{x_{j - \lambda}, \ldots, x_{\lambda - 1}\}$, for $\lambda \leq j < 2\lambda$.

**Proof.** Every element $x$ of $H_j$ can be written uniquely in the form $x = x_{i_1} x_{i_2} \cdots x_{i_r}$ with $0 \leq i_1 < i_2 < \cdots < i_r \leq j - 1$. Now $[x_{i_1}, x_{i_2 + 1}] \neq 1$ since otherwise $[x_0, x_\lambda] = 1$ and then $x_\lambda$ commutes with $x_0, x_1, \ldots, x_{\lambda - 1}$, so $H_{\lambda + 1} = \langle x_0, x_1, \ldots, x_\lambda \rangle$ is abelian, contradiction. Thus $x_{i_1 + 1} \in Z(H_j)$ when $i_1 + \lambda < j$. Similarly $[x_i, x_{i - \lambda}] \neq 1$ when $i - \lambda \geq 0$. Hence if $x \in Z(H_j)$ then $x = x_{i_1} x_{i_2} \cdots x_{i_r}$ where $i_1 \geq j - \lambda$ and $i_r < \lambda$.

Conversely, if $0 \leq j - \lambda \leq i < \lambda \leq j$ then $x_i$ commutes with all of $x_0, x_1, \ldots, x_{\lambda - 1}$, because $H_\lambda = \langle x_0, x_1, \ldots, x_{\lambda - 1} \rangle$ is abelian, and with all of $x_{\lambda}, \ldots, x_{j - 1}$, since $h^\lambda H_\lambda h^{-\lambda} = \langle x_{\lambda}, \ldots, x_{2\lambda - 1} \rangle$ is abelian (and $\lambda \leq j < 2\lambda$). Thus every such element $x_{i_1} x_{i_2} \cdots x_{i_r}$ is central in $H_j$.

**Lemma 4.5:** The derived subgroup of $H_{j + 1} = \langle x_0, x_1, \ldots, x_j \rangle$ is a subgroup of $\langle x_1, \ldots, x_{j - 1} \rangle$, for $1 \leq j \leq s - 2$.

**Proof.** Each of $\langle x_1, \ldots, x_j \rangle$ and $\langle x_0, \ldots, x_{j - 1} \rangle$ has index 2 in $H_{j + 1} = \langle x_0, x_1, \ldots, x_j \rangle$, and is therefore normal in $H_{j + 1}$. Their intersection $\langle x_1, \ldots, x_{j - 1} \rangle$ is a normal subgroup, of index 4, and (so) the quotient is abelian. Thus $\langle x_1, \ldots, x_{j - 1} \rangle$ contains all commutators of elements of $H_{j + 1}$, and hence contains the derived subgroup of $H_{j + 1}$.

Next, consider the element $[x_0, x_\lambda] = x_0^{-1} x_\lambda^{-1} x_0 x_\lambda = (x_0 x_\lambda)^2$. By Lemma 4.5, this lies in $\langle x_1, \ldots, x_{\lambda - 1} \rangle$, so can be written in the form $x_{i_1} \cdots x_{i_r}$, with $1 \leq i_1 < \cdots < i_r \leq \lambda - 1$.

We will take $\mu = i_1$ and $\nu = i_r$, and show that $\mu + \lambda \geq s - 1$ and $2\lambda - \nu \geq s - 1$, and hence that $\frac{2}{3}(s - 1) \leq \lambda$.

**Lemma 4.6:** If $[x_0, x_\lambda]$ is written as $x_{i_1} \cdots x_{i_r}$, with $0 < i_1 < \cdots < i_r < \lambda$, then

(a) $i_1 + \lambda \geq s - 1$, and (b) $2\lambda - i_r \geq s - 1$.

**Proof.** Take $\mu = i_1$ and $\nu = i_r$, so that $0 < \mu \leq \nu < \lambda$.

For (a), suppose that $\mu + \lambda \leq s - 2$. Then Lemma 4.5 implies that $[x_0, x_{\mu + \lambda}]$ lies in $\langle x_1, \ldots, x_{\mu + \lambda - 1} \rangle$, the centre of which is $\langle x_\mu, \ldots, x_\lambda \rangle$. The latter contains $x_\lambda$ and $x_\mu \cdots x_\nu = [x_0, x_\lambda]$, so both of these commute with $[x_0, x_{\mu + \lambda}]$. This gives
\[[x_0, x_\lambda]^{x_{\mu+\lambda}} = [x_0^{x_{\mu+\lambda}}, x_\lambda^{x_{\mu+\lambda}}] = [x_0^{x_{\mu+\lambda}}, x_\lambda] \\
= [x_0[x_0, x_{\mu+\lambda}], x_\lambda] = [x_0, x_{\mu+\lambda}]^{-1}x_\lambda^{-1}x_0[x_0, x_{\mu+\lambda}]x_\lambda \\
= [x_0, x_{\mu+\lambda}]^{-1}[x_0, x_\lambda]^{x_\lambda^{-1}}[x_0, x_{\mu+\lambda}]x_\lambda \\
= [x_0, x_{\mu+\lambda}]^{-1}[x_0, x_\lambda][x_0, x_{\mu+\lambda}] = [x_0, x_\lambda],
\]

and therefore \(x_\mu \ldots x_\nu = [x_0, x_\lambda]\) commutes with \(x_{\mu+\lambda}\), contradiction.

Similarly, if \(2\lambda - \nu \leq s - 2\) then \([x_0, x_{2\lambda-\nu}] \in \langle x_1, \ldots, x_{2\lambda-\nu-1} \rangle\), the centre of which is \(\langle x_\lambda, x_\nu, x_{2\lambda-\nu} \rangle\), so \([x_0, x_{2\lambda-\nu}]\) commutes with \(x_{\lambda-\nu}\) and \(x_{\mu+\lambda-\nu} \ldots x_\lambda = h^{\lambda-\nu}(x_\mu \ldots x_\nu)h^{\nu-\lambda} = h^{\lambda-\nu}[x_0, x_\lambda]h^{\nu-\lambda} = [x_{\lambda-\nu}, x_{2\lambda-\nu}].\) This gives
\[
[x_{\lambda-\nu}, x_{2\lambda-\nu}]^{x_0} = [x_{\lambda-\nu}^{x_0}, x_{2\lambda-\nu}^{x_0}] = [x_{\lambda-\nu}, x_{2\lambda-\nu}]
\]
\[
= [x_{\lambda-\nu}, [x_0, x_{2\lambda-\nu}]]x_{2\lambda-\nu} \\
= x_{\lambda-\nu}x_{2\lambda-\nu}[x_0, x_{2\lambda-\nu}]^{-1}x_{\lambda-\nu}[x_0, x_{2\lambda-\nu}]x_{2\lambda-\nu} \\
= x_{\lambda-\nu}x_{2\lambda-\nu}x_{\lambda-\nu}x_{2\lambda-\nu} = [x_{\lambda-\nu}, x_{2\lambda-\nu}],
\]
so \(x_{\mu+\lambda-\nu} \ldots x_\lambda = [x_{\lambda-\nu}, x_{2\lambda-\nu}]\) commutes with \(x_{\mu+\lambda}\), contradiction. This proves part (b).

Lemma 4.7: If \(s \geq 4\), then \(\lambda \geq \frac{2}{3}(s-1)\).

Proof. Lemma 4.6 gives \(s - 1 - \lambda \leq \mu \leq \nu \leq 2\lambda - s + 1\), and then forgetting \(\mu\) and \(\nu\) and rearranging gives \(2s - 2 \leq 3\lambda\).

Lemma 4.8: If \(s \geq 4\), then \(s = 4, 5\) or 7.

Proof. By Lemmas 3 and 6 we have \(\frac{2}{3}(s-1) \leq \lambda < \frac{1}{2}(s+2)\). Forgetting \(\lambda\) and rearranging gives \(4s - 4 < 3s + 6\), so \(s < 10\), but on the other hand, for \(s \in \{6, 8, 9\}\) there is no integer solution for \(\lambda\), so \(s = 4, 5\) or 7.

Lemma 4.9: \(s \neq 7\).

Proof. Assume that \(s = 7\). Then \(\lambda = 4\), and \(2 \leq \mu \leq \nu \leq 2\) so \(\mu = \nu = 2\), which gives \([x_0, x_4] = x_2\). Next we consider \([x_0, x_5]\). By Lemma 4.5, this lies in \(\langle x_1, x_2, x_3, x_4 \rangle\).

Suppose that \((x_0, x_5)^2 = [x_0, x_5]\) lies in \(\langle x_1, x_2, x_3 \rangle\). Then \(x_5x_0x_5\) lies in \(\langle x_0, x_1, x_2, x_3 \rangle = H_4\) and so fixes vertex \(v_3\) of our original 7-arc \(\sigma = (v_0, v_1, \ldots, v_7)\), and hence \(x_0\) fixes \(v_3^{x_5}\). Observe that \(x_0\) fixes \(v_0\) and \(v_1\), but \(v_2\) or \(v_3\), and so \(v_2^{x_5}\) is the third neighbour of \(v_1\), different from \(v_0\) and \(v_2\) but then also fixed by \(x_0\). It follows that \(x_0\) preserves the 7-arc \((v_0^{x_5}, v_2^{x_5}, v_1, v_2, v_3, v_4, v_5, v_6)\), contradiction.

Thus \([x_0, x_5] = y x_4\) for some \(y \in \langle x_1, x_2, x_3 \rangle\). In particular, \(y\) commutes with \(x_0\) and \(x_4\) (since \(\lambda = 4\)), and also \(y^2 = 1\) since \(\langle x_1, x_2, x_3 \rangle\) is abelian. But now it follows that
\[
x_2 = [x_0, x_4] = (x_0 x_4)^2 = (x_0 y x_4)^2 = (x_0 [x_0, x_5])^2 \\
= (x_5 x_0 x_5)^2 = x_5 x_0^2 x_5 = 1, \ a \ final \ contradiction.
\]

This completes the proof of Tutte's theorem.
1.5 Amalgams and Covers

Recall (from Section 3) that if $X$ is an arc-transitive graph, with automorphism group $G$, and $H$ is the stabiliser of a vertex $v$, then $X$ may be viewed as a double coset graph $X(G, H, D)$ where $D = HaH$ for some $a \in G$ (moving $v$ to a neighbour), with $a^2 \in H$.

**Lemma 5.1**: In the arc-transitive graph $X(G, H, HaH)$, the stabiliser in $G$ of the arc $(H, Ha)$ is the intersection $H \cap a^{-1}Ha$, and the stabiliser of the edge $(H, Ha)$ is the subgroup generated by $H \cap a^{-1}Ha$ and $a$. In particular, the valency of $X(G, H, HaH)$ is equal to the index $|H : H \cap a^{-1}Ha|$.

**Proof.** Let $v$ be the vertex $H$. Then since $a^2 \in H$, the element $a$ interchanges $v$ with its neighbour $w = v^a$, and hence reverses the arc $(v, w)$. Also $a^{-1}Ha$ is the stabiliser of the vertex $v^a = w$, so $H \cap a^{-1}Ha$ is the stabiliser of the arc $(v, w)$. The rest follows easily from this (and transitivity of $H$ on the neighbours of $v$). \qed

**Amalgams for symmetric graphs**

For the next part of this section, we will abuse notation and use $V$, $E$ and $A$ respectively for the stabilisers of the vertex $v$, the edge $(v, w)$, and the arc $(v, w)$, where $w$ is the neighbour of $v$ that is interchanged with $v$ by the arc-reversing automorphism $a$.

The triple $(V, E, A)$ may be called an *amalgam*. Note that $V \cap E = A$. Note also that if $X$ is connected, then $G = \langle HaH \rangle = \langle H, a \rangle = \langle H, H \cap a^{-1}Ha, a \rangle = \langle V, E \rangle$.

This amalgam specifies the kind of group action on $X$. For example, if $X$ is 3-valent and $V \cong S_3 \cong D_3$ and $E \cong V_4$ (the Klein 4-group), with $A = V \cap E \cong C_2$, then the action of $G$ on $X$ is 2-arc-regular, with the element $a$ being an involution (reversing the arc $(v, w)$).

Conversely, from any such triple $(V, E, A)$ we can form the amalgamated free product $\mathcal{U} = V *_A E$ (of the groups $V$ and $E$ with their intersection $A = V \cap E$ as amalgamated subgroup), which is a kind of *universal* group for such actions.

Specifically, $G$ is an arc-transitive group of automorphisms of the symmetric graph $X$, acting in the way that is specified, if and only if $G$ is a quotient of $\mathcal{U}$ via some homomorphism which preserves the amalgam (that is, preserves the orders of $V$, $E$ and $A$). When that happens, the homomorphism takes $V$, $E$ and $A$ (faithfully) to the stabilisers of some vertex $v$, incident edge $(v, w)$ and arc $(v, w)$ respectively.

This gives a way of classifying such graphs, or finding all of the examples of small order.

**Exercise 16**: What is the amalgam for the action of $S_3 \wr S_2$ on the graph $K_{3,3}$? Is this the same as the amalgam for the Petersen graph?

In 1980, Djoković and Miller determined all possible amalgams for an arc-transitive action of a group on a 3-valent graph with finite vertex-stabiliser. There are precisely seven such amalgams, which they called $1'$, $2'$, $2''$, $3'$, $4'$, $4''$ and $5'$. In each case, the given number is the value of $s$ for which the group acts regularly on $s$-arcs, and $'$ indicates that the group contains arc-reversing elements that are involutions (of order 2), while $''$ indicates that every arc-reversing element has order greater than 2. (Note that we require $a^2 \in H$ but not necessarily $a^2 = 1$.) The first examples of finite 3-valent graphs with full auto-
morphism groups of the types $2''$ and $4''$ were found by Conder and Lorimer (1989).

The universal groups for the seven Djoković-Miller amalgams are now customarily denoted by $G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2$ and $G_5$, with $s$ being the subscript, and with $G_s$ and $G_s^1$ corresponding to $s'$, and $G_s^2$ corresponding to $s''$.

For example, the group $G_1$ is the modular group $\langle h, a \mid h^3 = a^2 = 1 \rangle$, which is the free product of $V = \langle h \rangle \cong C_3$ and $E = \langle a \rangle \cong C_2$ (with $A = V \cap E = \{1\}$ amalgamated). Quotients of this group (in which the orders of $V$ and $E$ are preserved) are groups that act regularly on the arcs of a connected 3-valent symmetric graph.

Similarly, $G_2^1$ is the extended modular group $\langle h, p, a \mid h^3 = p^2 = a^2 = (hp)^2 = (ap)^2 = 1 \rangle$, which is the amalgamated free product of $V = \langle h, p \rangle \cong S_3$ and $E = \langle p, a \rangle \cong V_4$ with amalgamated subgroup $A = V \cap E = \langle p \rangle \cong C_2$, while on the other hand, $G_2^2$ is the group $\langle h, p, a \mid h^3 = p^2 = a^4 = (hp)^2 = a^2p = 1 \rangle$, which is the amalgamated free product of $V = \langle h, p \rangle \cong S_3$ and $E = \langle a \rangle \cong C_4$ with amalgamated subgroup $A = V \cap E = \langle p \rangle = \langle a^2 \rangle \cong C_2$.

The group $G_3$ has a presentation on five generators $h, p, q, r, s$ and $a$, obtainable from the amalgam $5'$, with $V = \langle h, p, q, r, s \rangle \cong S_4 \times C_2$ (of order 48), and $E = \langle p, q, r, s, a \rangle \cong D_4 \times V_4$, and $A = V \cap E = \langle p, q, r, s \rangle \cong D_4 \times C_2$ (of order 16). This was used by Conder (1988) to prove that for all but finitely many $n$, both $A_n$ or $S_n$ occur as the automorphism groups of 5-arc-transitive cubic graphs.

Relationships between these seven groups were investigated by Djoković and Miller (1980) and Conder and Lorimer (1989), and recently by Conder and Nedela (2009) to refine the Djoković-Miller classification of arc-transitive group actions on symmetric cubic graphs.

**Exercise 17:** Find an example of a symmetric cubic graph that admits actions of arc-transitive groups of types $1'$, $2'$, $2''$ and $3'$.

**Exercise 18:** Can you find an example of a symmetric cubic graph that admits an action by an arc-transitive group of type $3'$ but not one of type $1'$?

Also in 1987, Richard Weiss identified the amalgams for several different kinds of $s$-arc-transitive group actions on graphs of valency greater than 3.

For example, Weiss produced one that gives the universal group for 7-arc-transitive group actions on 4-valent graphs, with $V$ a group of order 11664 (being an extension of a group of order $3^5$ by $GL(2,3)$), and $E$ a group of order 5832, with $A = V \cap E$ having index 4 in $V$ and index 2 in $E$. This was used by Conder and Walker (1998) to prove the existence of infinitely many 7-arc-transitive 4-valent graphs (indeed with automorphism group $A_n$ or $S_n$, for all but finitely many $n$).

**Amalgams for semi-symmetric graphs**

The same kind of thing happens for semi-symmetric graphs. These can be analysed in terms of amalgams $(H, K, L)$ consisting of the stabilisers $H = G_v$ and $K = G_w$ of adjacent vertices and their intersection $L = G_v \cap G_w$, which is the stabiliser of the edge $\{v, w\}$, since semi-symmetric graphs are edge- but not arc-transitive.

For semi-symmetric 3-valent graphs, there are 15 different amalgams, determined by Goldschmidt (1980). These were used by Conder, Malnič, Marušič, Pisanski and Potočnik...
to find all semi-symmetric 3-valent graphs of small order (in 2001), and led to their discovery of the Ljubljana graph, which is a semi-symmetric 3-valent graph of order 112 with interesting properties.

**Graph quotients and covers**

If $X$ and $Y$ are graphs for which there exists a graph homomorphism from $Y$ onto $X$, then $X$ is called a *quotient* of $Y$, and if the homomorphism is locally bijective — that is, faithful on the neighbourhood of each vertex — then $Y$ is called a *cover* of $X$.

**Exercise 19**: Show that $K_4$ is a quotient of the cube graph $Q_3$. [Hint: antipodes!]

There are various ways of constructing covers of a given connected graph $X$. Some involve *voltage graph* techniques, which can be roughly described as follows:

Choose a spanning tree for $X$, and a permutation group $P$ on some set $\Omega$, and assign elements of $P$ to the co-tree edges (the edges not included in the spanning tree), with each such edge given a specific orientation, to make it an arc. Then take $|\Omega|$ identical copies of $X$, and for each co-tree arc $(v, w)$, use the label $\pi$ (from $P$) to define copies of the arc, from the vertex $v$ in the $j$th copy of $X$ to the vertex $w$ in the $(j\pi)$th copy of $X$, for all $j \in \Omega$. This gives a cover of $X$, with *voltage group* $P$.

**Exercise 20**: Construct $Q_3$ as a cover of $K_4$, using $P = S_2$.

In the 1970s, John Conway used a covering technique to produce infinitely covers of Tutte’s 8-cage, and hence prove (for the first time) that there are infinitely many finite 5-arc-transitive cubic graphs. [This is described in Biggs’s book *Algebraic Graph Theory*.]

Another way to construct covers of a symmetric graph $X$ is to use the universal group $\mathcal{U} = V \ast_A E$ associated with the action of some arc-transitive group $G$ of automorphisms of $X$. The group $G$ is a quotient $\mathcal{U}/K$ for some normal subgroup $K$ of $\mathcal{U}$, and then for any normal subgroup $L$ of $\mathcal{U}$ contained in $K$, the quotient $\mathcal{U}/L$ is an arc-transitive group of automorphisms of some cover of $X$.

### 1.6 Some Recent Developments

This final section describes a number of recent developments on topics mentioned earlier.

**Foster census**:

In the 1930s, Ronald M. Foster (a mathematician/engineer working for Bell Labs) began compiling a list of all known connected symmetric 3-valent graphs of order up to 512. This ‘census’ was published in 1988, and was remarkably good, with only a few gaps.

The Foster census was extended by Conder and Dobcsányi (2002), with the help of some computational group theory and distributed computing. The extended census filled the gaps in Foster’s list, and took it further, up to order 768. This also produced the smallest symmetric cubic graph of Djoković-Miller type 2”, on 448 vertices. (The previously smallest known example had order 665280.) In 2011/12, with the help of a new algorithm for finding finite quotients of finitely-presented groups, Conder extended this cen-
sus, to find all connected symmetric 3-valent graphs of order up to 10000.

**Other such lists:**

Primož Potočnik, Pablo Spiga and Gabriel Verret have developed new methods for finding all vertex-transitive cubic graphs of small orders, and in 2012 used these to find all such graphs of order up to 1280, as well as all symmetric 4-valent graphs of order up to 640 (using a relationship between these kinds of graphs).

In 2013, Conder and Potočnik extended the list of all semi-symmetric 3-valent graphs, up to order 10000 as well.

There are similar lists of arc transitive graphs embedded on surfaces as regular maps, with large automorphism group. See www.math.auckland.ac.nz/~conder for some of these.

**Related open problems (concerning pathological examples):**

(a) **What is the smallest symmetric cubic graph of Đoković-Miller type $4''$?**

Such a graph must be 4-arc-regular, but have no arc-reversing automorphisms of order 2. The smallest known example has order 5314410, with automorphism group an extension of $(C_3)^{11}$ by $\text{PGL}(2,9)$. There is another nice (but larger) example of order 20401920, with automorphism group the simple Mathieu group $M_{24}$.

(b) **What is the smallest 5-arc-transitive cubic graph $X$ with the property that its automorphism group is the only arc-transitive group of automorphisms of $X$?**

The smallest known example has order 2497430038118400, with automorphism group $M_{24} \wr S_2$. Examples are known with an alternating group $A_n$ as automorphism group, but the smallest such $n$ is 26.

(c) **What is the smallest half-arc-transitive graph $X$ for which the stabiliser of a vertex in $\text{Aut}X$ is neither abelian nor dihedral?**

Conder and Potočnik found one recently of order $90 \cdot 3^{10}$, with vertex-stabiliser $D_4 \times C_2$.

**Covers:**

Cheryl Praeger and some of her colleagues have done a lot of work on decomposing and constructing symmetric graphs via their quotients, and are using this to form a (loose) classification of all 2-arc-transitive finite graphs.

Group-theoretic covering methods have been applied to find all symmetric (or semisymmetric) regular covers of various small graphs, with abelian covering groups. For example:

- Cyclic symmetric coverings of $Q_3$ [Feng & Wang (2003)],
- Cyclic symmetric coverings of $K_{3,3}$ [Feng & Kwak (2004)],
- Elementary abelian symmetric covers of the Petersen graph [Malnič & Potočnik (2006)],
- Semisymmetric elementary abelian covers of the Möbius-Kantor graph [Malnič, Marušić, Miklavič & Potočnik (2007)],
- Elementary abelian symmetric covers of the Pappus graph [Oh (2009)],
- Elementary abelian symmetric covers of the octahedral graph [Kwak & Oh (2009)],
• Elementary abelian symmetric covers of $K_5$ [Kuzman (2010)].

In joint work with PhD student Jicheng Ma (2009–2012) we now have all symmetric abelian regular covers of $K_4$, $K_{3,3}$, $Q_3$, the Petersen graph and the Heawood graph.

**Degree-diameter problem:**

The degree-diameter problem involves finding the largest (regular) connected graph with given vertex-degree $d$ and diameter $D$; for example, the Petersen graph is the largest for $(d, D) = (3, 2)$. In his PhD thesis project (2005-2008), Eyal Loz used voltage graphs to find covers of various small vertex- and/or arc-transitive graphs that are now the best known graphs in over half of the cases in the degree-diameter table. For more information, see: moorebound.indstate.edu/wiki/The_Degree_Diameter_Problem_for_General_Graphs.

**Locally arc-transitive graphs:**

A semi-symmetric graph is not vertex-transitive, but nevertheless can have a high degree of symmetry (subject to that constraint). A graph $X$ is locally $s$-arc-transitive if the stabiliser in Aut$X$ of a vertex $v$ is transitive on all $s$-arcs in $X$ starting at $v$.

An unpublished theorem of Stellmacher (1996) states that *If $X$ is a finite locally s-arc-transitive graph, then $s \leq 9$.* Until recently, the only known examples for $s = 9$ came from classical generalised octagons and their covers. Such graphs are semi-symmetric (and hence bipartite) but not regular: vertices in different parts can have different valencies.

The smallest example for $s = 9$ has order 4680, with vertices of valency 3 in one part and 5 in the other. Its automorphism group is $^2F_4(2)$ (a Ree simple group), with vertex-stabilisers $H$ and $K$ of orders 12288 and 20480, and arc/edge-stabiliser $L = H \cap K$ of order 4096. In response to a comment by Michael Giudici at Rogla in 2011, Conder proved that the amalgamated free product $H \ast_L K$ has all but finitely many alternating groups $A_n$ as quotients. Hence there exist *infinitely locally 9-arc-transitive bipartite graphs with vertices of valency 3 in one part and 5 in the other*.

### 1.7 Selected References


Chapter 2

Imprimitive Permutation Groups

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SUMMARY

With the Classification of the Finite Simple Groups and the O’Nan-Scott Theorem, much detailed information concerning primitive permutation groups has now been obtained. While primitive permutation groups are interesting in their own right, primitive permutation groups are actually quite rare, with a “typical” transitive permutation group being imprimitive. However, primitive permutation groups are the building blocks of imprimitive permutation groups, and so are the building blocks of all transitive groups. We will discuss techniques to analyze imprimitive permutation groups (sometimes using the recently obtained detailed knowledge of primitive permutation groups), with an emphasis on determining information about the automorphism group of a vertex-transitive digraph.
2.1 Introduction

The O’Nan-Scott Theorem together with the Classification of the Finite Simple Groups is a powerful tool that gives the structure of all primitive permutation groups, as well as their actions. This has allowed for the solution to many classical problems, and has opened the door to a deeper understanding of imprimitive permutation groups, as primitive permutation groups are the building blocks of imprimitive permutation groups. We first give a more or less standard introduction to imprimitive groups, and then move to less well-known techniques, with an emphasis on studying automorphism groups of graphs.

A few words about these lecture notes. The lecture notes are an "expanded" version of the lecture - some of the lecture will be basically exactly these lecture notes, but in many cases the proofs of some background results (typically those that in my view are those whose proofs are primarily checking certain computations) are given in these lecture notes but will not be given in the lectures due to time constraints. Also, the material is organized into sections by topic, not by lecture.

2.2 Basic Results on Imprimitive Groups

Definition 2.2.1 Let $G$ be a transitive group acting on $X$. A subset $B \subseteq X$ is a block of $G$ if whenever $g \in G$, then $g(B) \cap B = \emptyset$ or $B$. If $B = \{x\}$ for some $x \in X$ or $B = X$, then $B$ is a trivial block. Any other block is nontrivial. If $G$ has a nontrivial block then it is imprimitive. If $G$ is not imprimitive, we say that $G$ is primitive. Note that if $B$ is a block of $G$, then $g(B)$ is also a block of $B$ for every $g \in G$, and is called a conjugate block of $B$. The set of all blocks conjugate to $B$, denoted $/B$, is a partition of $X$, and $/B$ is called a complete block system of $G$.

There does not seem to be a standard term for what is called here a complete block system of $G$. Other authors use a system of imprimitivity or a $G$-invariant partition for this term.

Theorem 2.2.2 Let $/B$ be a complete block system of $G$. Then every block in $/B$ has the same cardinality, say $k$. Further, if $m$ is the number of blocks in $/B$ then $mk$ is the degree of $G$.

Theorem 2.2.3 Let $G$ be a transitive group acting on $X$. If $N \triangleleft G$, then the orbits of $N$ form a complete block system of $G$.

Proof. Let $x \in X$ and $B$ the orbit of $N$ that contains $x$, so that $B = \{h(x) : h \in N\}$. Let $g \in G$, and for $h \in N$, denote by $h'$ the element of $N$ such that $gh = h'g$. Note $h'$ always exists as $N \triangleleft G$, and that $\{h' : h \in N\} = N$ as conjugation by $g$ induces an automorphism of $N$. Then $g(B) = \{gh(x) : h \in N\} = \{h'g(x) : h \in N\} = \{h(g(x)) : h \in N\}$. Hence $g(B)$ is the orbit of $N$ that contains $g(x)$, and as the orbits of $N$ form a partition of $X$, $g(B) \cap B = \emptyset$ or $B$. Thus $B$ is a block, and as every conjugate block $g(B)$ of $B$ is an orbit of $N$, the orbits of $N$ do indeed form a complete block system of $G$. □
Example 2.2.4 Define \( \rho, \tau : \mathbb{Z}_2 \times \mathbb{Z}_5 \to \mathbb{Z}_2 \times \mathbb{Z}_5 \) by \( \rho(i, j) = (i, j + 1) \) and \( \tau(i, j) = (i + 1, 2j) \). Note that in these formulas, arithmetic is performed modulo 2 in the first coordinate and modulo 5 in the second coordinate. It is straightforward but tedious to check that \( \langle \rho, \tau \rangle \) is a subgroup of the automorphism group of the Petersen graph with the labeling shown in Figure 2.1.

![Figure 2.1: The Petersen graph.](image)

Additionally, \( \tau^{-1}(i, j) = (i - 1, 3j) \) as

\[
\tau^{-1} \tau(i, j) = \tau^{-1}(i + 1, 2j) = (i + 1 - 1, 3(2j)) = (i, j).
\]

Also,

\[
\tau^{-1} \rho \tau(i, j) = \tau^{-1}(i + 1, 2j + 1) = (i + 1 - 1, 3(2j + 1)) = (i, j + 3) = \rho^3(i, j)
\]

and so \( \langle \rho \rangle \triangleleft \langle \rho, \tau \rangle \). Then by Theorem 2.2.3 the orbits of \( \langle \rho \rangle \), which are the sets \{\{i, j\} : j \in \mathbb{Z}_5\} : i \in \mathbb{Z}_2\} form a complete block system of \( \langle \rho, \tau \rangle \).

Although we will not show this here, the full automorphism group of the Petersen graph is primitive.

A complete block system of \( G \) formed by the orbits of normal subgroup of \( G \) is called a **normal complete block system** of \( G \). Note that not every complete block system \( \mathcal{B} \) of every transitive group \( G \) is a complete block system of \( G \), as we shall see.

Now suppose that \( G \leq S_n \) is a transitive group which admits a complete block system \( \mathcal{B} \) consisting \( m \) blocks of size \( k \). Then \( G \) has an **induced action on** \( \mathcal{B} \), which we denote by \( G/\mathcal{B} \). Namely, for specific \( g \in G \), we define \( g/\mathcal{B}(B) = B' \) if and only if \( g(B) = B' \), and set \( G/\mathcal{B} = \{g/\mathcal{B} : g \in G\} \). We also define the **fixer of** \( \mathcal{B} \) **in** \( G \), denoted \( \text{fix}_G(\mathcal{B}) \), to be \( \{g \in G : g/\mathcal{B} = 1\} \). That is, \( \text{fix}_G(\mathcal{B}) \) is the subgroup of \( G \) which fixes each block of \( \mathcal{B} \) set-wise. Furthermore, \( \text{fix}_G(\mathcal{B}) \) is the kernel of the induced homomorphism \( G \to S_\mathcal{B} \), and as such is normal in \( G \). Additionally, \( |G| = |G/\mathcal{B}| \cdot |\text{fix}_G(\mathcal{B})| \).

A transitive group \( G \) is **regular** if \( \text{Stab}_G(x) = 1 \) for any (and so all) \( x \).
Theorem 2.2.5 Let \( G \leq S_n \) be transitive with an abelian regular subgroup \( H \). Then any complete block system of \( G \) is normal, and is formed by the orbits of a subgroup of \( H \).

**Proof.** We only need show that \( \text{fix}_H(\mathcal{B}) \) has orbits of size \( |B| \), \( B \in \mathcal{B} \). Now, \( H/\mathcal{B} \) is transitive and abelian, and so \( H/\mathcal{B} \) is regular (a transitive abelian group is regular as conjugation permutes the stabilizers of points - so in a transitive abelian group, point stabilizers are all equal). Then \( H/\mathcal{B} \) has degree \( |\mathcal{B}| \), and so there exists nontrivial \( K \leq \text{fix}_H(\mathcal{B}) \) of order \( |B| \). Then the orbits of \( K \) form a complete block system \( \mathcal{C} \) of \( H \) by Theorem 2.2.3, and each block of \( \mathcal{C} \) is contained in a block of \( \mathcal{B} \). As \( K \) has order \( |B| \), we conclude that \( \mathcal{C} = \mathcal{B} \). \( \square \)

Lemma 2.2.6 Let \( G \) act transitively on \( X \), and let \( x \in X \). Let \( H \leq G \) be such that \( \text{Stab}_G(x) \leq H \). Then the orbit of \( H \) that contains \( x \) is a block of \( G \).

**Proof.** Set \( B = \{h(x) : h \in H\} \) (so that \( B \) is the orbit of \( H \) that contains \( x \)), and let \( g \in G \).

We must show that \( B \) is a block of \( G \), or equivalently, that \( g(B) = B \) or \( g(B) \cap B = \emptyset \). Clearly if \( g \in H \), then \( g(B) = B \) as \( B \) is the orbit of \( H \) that contains \( x \) and \( x \in B \). If \( g \notin H \), then towards a contradiction suppose that \( g(B) \cap B \neq \emptyset \), with say \( z \in g(B) \cap B \). Then there exists \( y \in B \) such that \( g(y) = z \) and \( h, k \in H \) such that \( h(x) = y \) and \( k(x) = z \). Then

\[
z = g(y) = gh(x) = k(x) = z,
\]

and so \( gh(x) = k(x) \). Thus \( k^{-1}gh \in \text{Stab}_G(x) \). This then implies that \( g \in k \cdot \text{Stab}_G(x) \cdot h^{-1} \leq H \), a contradiction. Thus if \( g \notin H \), then \( g(B) \cap B = \emptyset \), and \( B \) is a block of \( G \). \( \square \)

Example 2.2.7 Consider the subgroup of the automorphism group of the Petersen graph \( \langle \rho, \tau \rangle \) that we saw before. Straightforward computations will show that \( |\tau| = 4 \), and so \( |\langle \rho, \tau \rangle| = 20 \) as \( |\rho| = 5 \). By the Orbit-Stabilizer Theorem, we have that \( \text{Stab}_{\langle \rho, \tau \rangle}(0,0) \) has order 2, and as \( \tau^2 \) stabilizes \( (0,0) \), \( \text{Stab}_{\langle \rho, \tau \rangle}(0,0) = \langle \tau^2 \rangle \). Then \( \langle \tau \rangle \leq \langle \rho, \tau \rangle \) and contains \( \text{Stab}_{\langle \rho, \tau \rangle}(0,0) \). Then the orbit of \( \langle \tau \rangle \) that contains \( (0,0) \) is a block of \( \langle \rho, \tau \rangle \) as well. This orbit is \( \{(0,0), (1,0)\} \). So the corresponding complete block system of \( \langle \rho, \tau \rangle \) consists of the vertices of the “spoke” edges of the Petersen graph.

Just as we may examine the stabilizer of a point in a transitive group \( G \), we may also examine the stabilizer of the block \( B \) in an imprimitive group \( G \). It is denoted \( \text{Stab}_G(B) \), is a subgroup of \( G \), and \( \text{Stab}_G(B) = \{g \in G : g(B) = B\} \).

Theorem 2.2.8 Let \( G \) act transitively on \( X \), and let \( x \in X \). Let \( \Omega \) be the set of all blocks \( B \) of \( G \) which contain \( x \), and \( S \) be the set of all subgroups \( H \leq G \) that contain \( \text{Stab}_G(x) \). Define \( \phi : \Omega \to S \) by \( \phi(B) = \text{Stab}_G(B) \). Then \( \phi \) is a bijection, and if \( B, C \in \Omega \), then \( B \subseteq C \) if and only if \( \text{Stab}_G(B) \leq \text{Stab}_G(C) \).

**Proof.** First observe that \( \text{Stab}_G(x) \leq \text{Stab}_G(B) \) for every block \( B \) with \( x \in B \), so \( \phi \) is indeed a map from \( \Omega \) to \( S \). We first show that \( \phi \) is onto. Let \( H \in S \) so that \( \text{Stab}_G(x) \leq H \). By Lemma 2.2.6, \( B = \{h(x) : h \in H\} \) is a block of \( G \). Then \( H \leq \phi(B) \). Towards a contradiction, suppose there exists \( g \in \phi(B) \) such that \( g \notin H \). Then \( g(B) = B \), and \( H \) is transitive in its action on \( B \) (Exercise 2.2.12). Hence there exists \( h \in H \) such that \( h(x) = g(x) \), and so
Let \( G \) be a transitive group acting on \( X \). If \( \equiv \) is an equivalence relation on \( X \) such that \( x \equiv y \) if and only if \( g(x) \equiv g(y) \) for all \( g \in G \) (a \( G \)-congruence), then the equivalence classes of \( \equiv \) form a complete block system of \( G \).

**Proof.** Let \( B_x \) be an equivalence class of \( \equiv \) that contains \( x \), and \( x \in X \), \( g \in G \). Then

\[
g(B_x) = \{g(y) : y \in X \text{ and } x \equiv y\} = \{g(y) : g(y) \equiv g(x)\} = B_{g(x)}.
\]

As the equivalence classes of \( \equiv \) form a partition of \( X \), it follows that \( g(B_x) \cap B_x = \emptyset \) or \( B_x \), and so \( B_x \) is a block of \( G \). Also, as \( g(B_x) = B_{g(x)} \), the set of all blocks conjugate to \( B_x \) is just the set of equivalence classes of \( \equiv \). \( \square \)

A common application of the above result is to stabilizers of points, as in a transitive group, any two point stabilizers are conjugate (Exercise ?).

**Exercise 2.2.10** Verify that if \( B \) is a block of \( G \), then \( g(B) \) is also a block of \( G \) for every \( g \in G \).

**Exercise 2.2.11** Verify that if \( \mathcal{B} \) is a complete block system of \( G \) acting on \( X \), then \( \mathcal{B} \) is a partition of \( X \).

**Exercise 2.2.12** Let \( G \) act transitively on \( X \), and suppose that \( B \) is a block of \( G \). Then \( \text{Stab}_G(B) \) is transitive on \( B \).

**Exercise 2.2.13** Show that a transitive group of prime degree is primitive.

**Exercise 2.2.14** Let \( G \leq S_n \) with \( \mathcal{B} \) a complete block system of \( G \). If \( \phi \in S_n \), then \( \phi(\mathcal{B}) \) is a \( \phi G \phi^{-1} \)-invariant partition.

**Exercise 2.2.15** A group \( G \) acting on \( X \) is doubly-transitive if whenever \((x_1, y_1), (x_2, y_2) \in X \times X \) such that \( x_1 \neq y_1 \) and \( x_2 \neq y_2 \), then there exists \( g \in G \) such that \( g(x_1, y_1) = (x_2, y_2) \). Show that a doubly-transitive group is primitive.
Exercise 2.2.16 Let $G \leq S_n$ contain a regular cyclic subgroup $R = \langle 0 \ 1 \ \ldots \ n-1 \rangle$ and admit a complete block system $\mathcal{B}$ consisting of $m$ blocks of size $k$. Show that $\mathcal{B}$ consists of cosets of the unique subgroup of $\mathbb{Z}_n$ of order $k$.

Exercise 2.2.17 Let $p$ and $q$ be distinct primes such that $q$ divides $p-1$. Determine the number of complete block systems of $G_1$ where $G$ is the nonabelian group of order $pq$ that consist of blocks of cardinality $q$ and of cardinality $p$.

Exercise 2.2.18 Let $G$ be a transitive group of square-free degree (an integer that is square-free is not divisible by the square of any prime). Show that $G$ has at most one normal $G$-invariant partition with blocks of prime size $p$. (Hint: Suppose there are at least two such $G$-invariant partitions $\mathcal{B}_1$ and $\mathcal{B}_2$. Consider what happens to $\text{fix}_G(\mathcal{B}_2)$ in $G/\mathcal{B}_1$.)

Exercise 2.2.19 Let $G \leq S_n$ be transitive. Show that $G$ is primitive if and only if $\text{Stab}_G(x)$ is a maximal subgroup of $G$ for every $x \in \mathbb{Z}_n$.

2.3 Notions of “Sameness” of Permutation Groups

First, a group $G$ may be represented as a permutation group in different ways. We first need to be able to distinguish when two such representations are essentially the same, or are different.

Definition 2.3.1 We say that the action of $G$ on sets $A$ and $B$ are permutation equivalent if there exists a bijection $\lambda : A \to B$ such that $\lambda(g(x)) = g(\lambda(x))$ for all $x \in A$ and $g \in G$.

On an obvious way in which $G$ can have equivalent actions on different sets $A$ and $B$ are if $|A| = |B|$, and we simply relabel the elements of $A$ with elements of $B$. In this case, the defining equation $\lambda(g(x)) = g(\lambda(x))$ for all $x \in A$ and $g \in G$ states that if we apply $g$ to $x$ and then relabel, then this is the same as if we relabel $x$ and then apply $g$ to the relabelled $x$. For our purposes, we will be concerned with transitive groups. The following result is used in practice to determine if two actions of $G$ are equivalent.

Theorem 2.3.2 Let $G$ act transitively on $A$ and $B$. Then the action of $G$ on $A$ is equivalent to the action of $G$ on $B$ if and only if the stabilizer in $G$ of a point in $A$ is the stabilizer of a point in $B$.

Proof. Suppose that the action of $G$ on $A$ is equivalent to the action of $G$ on $B$. Then there exists a bijection $\lambda : A \to B$ such that $\lambda(g(x)) = g(\lambda(x))$ for all $x \in A$ and $g \in G$. Let $K = \text{Stab}_G(z)$, where $z \in B$, and $y \in A$ such that $\lambda(y) = z$. Let $k \in K$. As $k(z) = z$, we have that

$$\lambda(k(y)) = k(\lambda(y)) = k(z) = z.$$ 

As $\lambda$ is a bijection, $k(y) = \lambda^{-1}(z) = y$, and so $k$ stabilizes $y$. Thus $K \leq \text{Stab}_G(y)$, and as $G$ is transitive on $A$ and $B$ and $|A| = |B|$, by the Orbit-Stabilizer Theorem we see that $K = \text{Stab}_G(y)$.

Now suppose that $\text{Stab}_G(a) = \text{Stab}_G(b)$ for some $a \in A$ and $b \in B$. Define $\lambda : A \to B$ by $\lambda(g(a)) = g(b)$. We first need to show that $\lambda$ is well-defined. That is, that regardless of
choice of \(g, \lambda(x) = y, x \in A, y \in B\), is the same. So we need to show that if \(g(a) = h(a)\), then \(g(b) = \lambda(g(a)) = \lambda(h(a)) = h(b)\). Now,

\[
\begin{align*}
g(a) &= h(a) \Rightarrow h^{-1}g(a) = a \\
&\quad \Rightarrow h^{-1}g \in \text{Stab}_G(a) \\
&\quad \Rightarrow h^{-1}g(b) = b \\
&\quad \Rightarrow g(b) = h(b) \\
&\quad \Rightarrow \lambda(g(a)) = \lambda(h(a))
\end{align*}
\]

and so \(\lambda\) is indeed well-defined. Also, as \(G\) is transitive on \(A\), \(\lambda\) has domain \(A\), and as \(G\) is transitive on \(B\), \(\lambda\) is surjective, and hence bijective. Finally, let \(x \in A\). Then there exists \(h_x \in G\) such that \(h_x(a) = x\). Then

\[
\begin{align*}
\lambda(g(x)) &= \lambda(g h_x(a)) \\
&= g h_x(b) \\
&= g \lambda(h_x(a)) \\
&= g \lambda(x).
\end{align*}
\]

**Definition 2.3.3** Let \(G \leq S_A\) and \(H \leq S_B\). Then \(G\) and \(H\) are **permutation isomorphic** if there exists a bijection \(\lambda : A \to B\) and a group isomorphism \(\phi : G \to H\) such that \(\lambda(g(x)) = \phi(g)(\lambda(x))\) for all \(x \in A\) and \(g \in G\).

Intuitively, in addition to relabeling the set on which \(G\) acts (via \(\lambda\)), we also relabel the group (via the homomorphism \(\phi\)).

**Theorem 2.3.4** Let \(G\) be a transitive group acting on \(A\) that admits a complete block system \(\mathcal{B}\). Then the action of \(\text{Stab}_G(B)\) on \(B\) and the action of \(\text{Stab}_G(B')\) on \(B'\) are permutation isomorphic. Additionally, the action of \(\text{fix}_G(\mathcal{B})\) on \(B\) is permutation isomorphic to the action of \(\text{fix}_G(\mathcal{B})\) on \(B'\).

**Proof.** Let \(\ell \in G\) such that \(\ell(B) = B'\). Define \(\lambda : B \to B'\) by \(\lambda(x) = \ell(x)\). As \(\ell\) maps \(B\) bijectively to \(B'\), \(\lambda\) is a bijection. Define \(\phi : \text{Stab}_G(B) \to \text{Stab}_G(B')\) by \(\phi(g) = \ell g \ell^{-1}\). As \(\phi\) is obtained by conjugation, \(\phi\) is a group isomorphism. Let \(g \in \text{Stab}_G(B)\), and \(x \in B\). Then

\[
\begin{align*}
\lambda(g(x)) &= \ell g(x) = \ell g \ell^{-1} \ell(x) = \phi(g) \lambda(x),
\end{align*}
\]

and so the action of \(\text{Stab}_G(B)\) on \(B\) is permutation isomorphic to the action of \(\text{Stab}_G(B')\) on \(B'\). Analogous arguments will show that the action of \(\text{fix}_G(\mathcal{B})\) on \(B\) is permutation isomorphic to the action of \(\text{fix}_G(\mathcal{B})\) on \(B'\). \(\square\)
2.4 An Example of Inequivalent Actions: The Automorphism Group of the Heawood Graph

For a subspace $S$ of $\mathbb{F}_q^n$, we denote by $S^\perp$ the orthogonal complement of $S$. That is, $S^\perp = \{w \in \mathbb{F}_q^n : w \cdot v = 0 \text{ for every } v \in S\}$. Recall that $S^\perp$ is a subspace of the vector space $\mathbb{F}_q^n$. A line in $\mathbb{F}_q^n$ is a one-dimensional subspace, while a hyperplane is the orthogonal complement of a line (so a subspace of $\mathbb{F}_q^n$ of dimension $n-1$). Note that the number of lines and hyperplanes of $\mathbb{F}_q^n$ are the same. In the case of $\mathbb{F}_3^2$ which contains 8 elements, any nonzero vector gives rise to a line, so there are 7 lines and 7 hyperplanes.

Consider the graph whose vertex set is the lines and hyperplanes of $\mathbb{F}_3^2$, and a line is adjacent to a hyperplane if and only if the line is contained in the hyperplane. We obtain the following graph, which is isomorphic to the Heawood graph:

Figure 2.2: The Heawood graph labeled with the lines and hyperplanes of $\mathbb{F}_3^2$

Recall that $\text{GL}(n, \mathbb{F}_q)$ is the general linear group of dimension $n$ over the field $\mathbb{F}_q$. That is $\text{GL}(n, \mathbb{F}_q)$ is the group of all invertible $n \times n$ matrices with entries in $\mathbb{F}_q$, with binary operation multiplication. In the literature, it is common to see $\text{GL}(n, q)$ written in place of $\text{GL}(n, \mathbb{F}_q)$, a convention that we will follow. Of course, a linear transformation maps lines to lines, so we can consider the action of $\text{GL}(3, 2)$ on the lines of $\mathbb{F}_3^2$, and obtain the group $\text{PGL}(3, 2)$, which is isomorphic to $\text{GL}(3, 2)$. Note that $\text{PGL}(3, 2)$ also permutes the hyperplanes of $\mathbb{F}_3^2$. Of course, an element of $\text{PGL}(3, 2)$ maps a line contained in a hyperplane to a line contained in a hyperplane, and so $\text{PGL}(3, 2)$ is contained in $\text{Aut}(\text{Hea})$, where $\text{Hea}$ is the Heawood graph. Notice that $\text{PGL}(3, 2)$ is transitive on the lines of $\mathbb{F}_3^2$ and transitive on the hyperplanes of $\mathbb{F}_3^2$.

Now define $\tau : L \cup H \to L \cup H$ by $\tau \{\ell, h\} = \{h^\perp, \ell^\perp\}$. Note that $\tau$ is well-defined, as the subspace orthogonal to a line is a hyperplane, while the subspace orthogonal to a hyperplane is a line. Clearly $|\tau| = 2$ as $(s^\perp)^\perp = s$. In order to show that $\tau \in \text{Aut}(\text{Hea})$, let $\ell \in L$ and $h \in H$ such that $\ell \subset h$. Then every vector in $h^\perp$ is orthogonal to every vector in $h$, and as $\ell \subset h$, every vector in $h^\perp$ is orthogonal to every vector in $\ell$. Thus $h^\perp \subset \ell^\perp$ and
Lemma 2.4.1 Let $g \in \text{GL}(n, q)$, and $s$ a subspace of $\mathbb{F}_q^n$. Then $g(s^\perp) = (g^{-1})^T(s)$.

Proof. First recall that if $w, v \in \mathbb{F}_q^n$, then the dot product of $w$ and $v$, $w \cdot v$, can also be written as $w^T v$, where for a matrix $g$, $g^T$ denotes the transpose of $g$. Let $w_1, \ldots, w_r$ be a basis for $s^\perp$, so that $g(s^\perp)$ has basis $gw_1, \ldots, gw_r$. In order to show that $g(s^\perp) = (g^{-1})^T(s)$, it suffices to show that $(g^{-1})^T v$ is orthogonal to $gw_i$ for any $i$ and $v \in s$ as $\dim(s) + \dim(s^\perp) = n$. Then

$$\langle gw_i, (g^{-1})^T v \rangle = \langle gw_i, g^{-1} g^T (g^{-1})^T v \rangle = \langle w_i, g^{-1} g^T (g^{-1})^T v \rangle = \langle w_i, v \rangle = 0.$$ 

□

Consider the canonical action of $\text{PGL}(3, 2)$ on $L \cup H$, so that $g \in \text{PGL}(3, 2)$, then $g(\ell, h) = \{g(\ell), g(h)\}$. Now, let $g \in \text{PGL}(3, 2)$, which will consider in the above action on $L \cup H$. Then

$$\tau^{-1} g \tau(\{\ell, h\}) = \tau^{-1} g(\{h^\perp, \ell^\perp\})$$
$$= \tau^{-1}(\{g(h^\perp), g(\ell^\perp)\})$$
$$= \{g(\ell^\perp), g(h^\perp)\}$$
$$= \{(g^{-1})^T(\ell), (g^{-1})^T(h)\}$$

Then $\tau^{-1} g \tau = (g^{-1})^T$ so $\text{PGL}(3, 2) \wr \langle \text{PGL}(3, 2), \tau \rangle$.

Now, $(\text{PGL}(3, 2), \tau)$ admits a complete block system $\mathcal{B}$ with 2 blocks of size 7. The subgroup of $\text{PGL}(3, 2)$ that stabilizes a line does not stabilize any hyperplane! So we have that $\text{PGL}(3, 2)$ acts inequivalently on the lines and hyperplanes. It can be shown using a theorem of Tutte that $\text{Aut(Hea)} = \langle \text{PGL}(3, 2), \tau \rangle$.

### 2.5 The Embedding Theorem

Definition 2.5.1 Let $\Gamma_1$ and $\Gamma_2$ be digraphs. The **wreath product of $\Gamma_1$ and $\Gamma_2$**, denoted $\Gamma_1 \wr \Gamma_2$ is the digraph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and edge set

$$\{(u, v)(u', v') : u \in V(\Gamma_1) \text{ and } v v' \in E(\Gamma_2) \cup \{(u, v)(u', v') : u u' \in E(\Gamma_1) \text{ and } v, v' \in V(\Gamma_2)\}.$$

Intuitively, $\Gamma_1 \wr \Gamma_2$ is constructed as follows. First, we have $|V(\Gamma_1)|$ copies of the digraph $\Gamma_2$, with these $|V(\Gamma_1)|$ copies indexed by elements of $V(\Gamma_1)$. Next, between corresponding copies of $\Gamma_2$ we place every possible directed from one copy to another if in $\Gamma_1$ there is an edge between the indexing labels of the copies of $\Gamma_2$, and no edges otherwise.

To find the wreath product of any two graphs $\Gamma_1$ and $\Gamma_2$ (see Figure 2.3):

1. First corresponding to each vertex of $\Gamma_1$, put a copy of $\Gamma_2$.
2. If $v_1$ and $v_2$ are adjacent in $\Gamma_1$, put every edge between corresponding copies of $\Gamma_2$. 
Let us consider the graph $C_8 \wr \bar{K}_2$ (see Figure 2.5).

In the previous graph, think of the sets $\{(i,j) : j \in \mathbb{Z}_2\}$ as blocks. Take any automor-
phism of $C_8$, and think of it as “permuting” the blocks. A block is mapped to a block by any automorphism of $K_2$, and we can have different automorphisms of $K_2$ for different blocks. This is the group $\text{Aut}(C_8) \wr \text{Aut}(K_2)$.

**Definition 2.5.2** Let $G$ be a permutation group acting on $X$ and $H$ a permutation group acting on $Y$. Define the *wreath product of $G$ and $H$*, denoted $G \wr H$, to be the set of all permutations of $X \times Y$ of the form $(x, y) \mapsto (g(x), h_x(y))$.

Intuitively, the wreath product $G \wr H$ has elements of $G$ permuting $|X|$ copies of $Y$, and as an element of $G$ permutes these copies, the copies of $Y$ are mapped to each via elements of $H$. Crucially, the elements of $H$ chosen to map copies of $Y$ mapped to each other are chosen independently.

**Example 2.5.3** We show the group $\mathbb{Z}_p \wr \mathbb{Z}_p \leq S_{p^2}$ has order $p^{p+1}$ and consequently is a Sylow $p$-subgroup of $S_{p^2}$. As $\mathbb{Z}_p \wr \mathbb{Z}_p = \{(i, j) \mapsto (i + a, j + b) : a, b \in \mathbb{Z}_p\}$, we see that $|\mathbb{Z}_p \wr \mathbb{Z}_p| = p^{p+1}$ as there are $p$ choices for $a$, as well as $p$ choices for each of the $p$ $b_i$. As the multiples of $p$ dividing $p^2$ are $p, 2p, \ldots, (p - 1)p, p^2$, we see that the largest power of $p$ dividing $p^2!$ is $p^{p+1}$.

**Definition 2.5.4** Let $G$ be a transitive permutation group acting on $X$ that admits a complete block system $\mathcal{B}$. Then the action of $G$ on $\mathcal{B}$ induces a permutation group in $S_{\mathcal{B}}$, which we denote by $G/\mathcal{B}$. More specifically, if $g \in G$, then define $g/\mathcal{B} : \mathcal{B} \to \mathcal{B}$ by $g/\mathcal{B}(B) = B'$ if and only if $g(B) = B'$, and set $G/\mathcal{B} = \{g/\mathcal{B} : g \in G\}$.

**Theorem 2.5.5** Let $G$ be a transitive permutation group acting on $X$ that admits a complete block system $\mathcal{B}$. Then $G$ is permutation isomorphic to a subgroup of

$$(G/\mathcal{B}) \wr (\text{Stab}_G(\mathcal{B}))_{B_0},$$

where $B_0 \in \mathcal{B}$.

**Proof.** For each $B \in \mathcal{B}$, there exists $h_B$ such that $h_B(B_0) = B$. Define $\lambda : X \to \mathcal{B} \times B_0$ by $\lambda(x) = (B, x_0)$, where $x \in B$ and $x_0 = h_B^{-1}(x)$. Define $\phi : G \to (G/\mathcal{B}) \wr (\text{Stab}_G(\mathcal{B}))_{B_0} \to (G/\mathcal{B}, g^{-1}_B g h_B(x_0))$. We must show that $\lambda$ is a bijection, $\phi$ is an injective homomorphism, and $\phi(g(x)) = \phi(g)(\lambda(x))$ for all $x \in X$ and $g \in G$.

In order to show that $\lambda$ is a bijection, it suffices to show that $\lambda$ is one-to-one as by Theorem 2.2.2 it is certainly the case that $|X| = |\mathcal{B} \times B_0|$. Let $x, x' \in X$ and assume that $(B, x_0) = \lambda(x) = \lambda(x')$. Clearly then both $x$ and $x'$ are contained in $B$, and $x_0 = h_B^{-1}(x) = h_B^{-1}(x')$. As $h_B$ is a permutation, it follows that $x = x'$ and $\lambda$ is one-to-one and so a bijection.

To show that $\phi$ is injective, suppose that $\phi(g) = \phi(g')$. Applying the definition of $\phi$, we see that

$$(g(B), h^{-1}_B g h_B(x_0)) = (g'(B), h^{-1}_B g' h_B(x_0)),$$

for all $B \in \mathcal{B}$ and $x_0 \in B_0$. It immediately follows that $g/\mathcal{B} = g'/\mathcal{B}$ and $h^{-1}_B g h_B = h^{-1}_B g' h_B$. Using the fact that $g/\mathcal{B} = g'/\mathcal{B}$ and canceling, we see that $g = g'$ and $\phi$ is injective.
Let \( g_1, g_2 \in G \). Then
\[
\phi(g_1)\phi(g_2)(B, x_0) = \phi(g_1)(g_2(B), h_{g_2^{-1}(B)}g_2 h_B(x_0)) \\
= (g_1 g_2(B), h_{g_2(B)}^{-1} g_1 h_{g_2(B)} g_2 h_B(x_0)) \\
= \phi(g_1 g_2)(B, x_0),
\]
and so \( \phi \) is a homomorphism.

Finally, observe that \( \phi(g)(\lambda(x)) = \phi(g)(B, x_0) = (g(B), h_{g(B)}^{-1} g h_B(x_0)) \) while
\[
\lambda(g(x)) = (g(B), h_{g(B)}^{-1} g h_B(x_0)) = (g(B), h_{g(B)}^{-1} g h_B(x_0)),
\]
and so \( \lambda(g(x)) = \phi(g)(\lambda(x)) \) for all \( x \in X \) and \( g \in G \).

The following immediate corollary is often useful.

**Corollary 2.5.6** Let \( G \) be a transitive permutation group that admits a complete block system \( \mathcal{B} \) consisting of \( m \) blocks of size \( k \). Then \( G \) is permutation isomorphic to a subgroup of \( S_m \wr S_k \).

One must be slightly careful with this labeling, as it is not always the most natural labeling. For example, let \( q \) and \( p \) be prime with \( q | (p - 1) \) and \( \alpha \in \mathbb{Z}_p^* \) of order \( q \). Define \( \rho, \tau : \mathbb{Z}_q \times \mathbb{Z}_p \rightarrow \mathbb{Z}_q \times \mathbb{Z}_p \) by \( \tau(i, j) = (i + 1, \alpha j) \) and \( \rho(i, j) = (i, j + 1) \). Then \( \langle \rho, \tau \rangle \) is isomorphic to the nonabelian group of order \( qp \). The labeling that one would get for this group by applying the Embedding Theorem is \( \langle \rho', \tau' \rangle \), where \( \rho'(i, j) = (i, j + \alpha^i) \), \( \tau'(i, j) = (i + 1, j) \).

**Exercise 2.5.7** Draw the graph \( K_4 \wr \bar{K}_3 \).

**Exercise 2.5.8** Show that \( G \wr H \) has order \( |G| \cdot (|H|)^{|X|} \), where \( G \) acts on \( X \).

**Exercise 2.5.9** Show that a Sylow \( p \)-subgroup of \( S_{p^k}, k \geq 1 \) is \( \mathbb{Z}_p \wr \mathbb{Z}_p \wr \cdots \wr \mathbb{Z}_p \), where the wreath product is taken \( k \) times.

**Exercise 2.5.10** Verify that the graph wreath product is associative.

**Exercise 2.5.11** Verify that the permutation group wreath product is associative.

**Exercise 2.5.12** Show that \( \text{Aut}(\Gamma_1 \wr \Gamma_2) = \text{Aut}(\Gamma_1) \wr \text{Aut}(\Gamma_2) \).

**Exercise 2.5.13** For vertex-transitive graphs \( \Gamma_1 \) and \( \Gamma_2 \), show that \( \text{Aut}(\Gamma_1) \wr \text{Aut}(\Gamma_2) \leq \text{Aut}(\Gamma) \).
2.6 A Graph Theoretic Tool

Let $G$ be a transitive group that admits a normal complete block system $\mathcal{B}$ consisting of $m$ blocks of prime size $p$. Then $\text{fix}_G(\mathcal{B})|_\mathcal{B}$ is a transitive group of prime degree $p$, and so contains a $p$-cycle. Define a relation $\equiv$ on $\mathcal{B}$ by $B \equiv B'$ if and only if whenever $\gamma \in \text{fix}_G(\mathcal{B})$ then $\gamma|_B$ is a $p$-cycle if and only if $\gamma|_{B'}$ is also a $p$-cycle (here $\gamma|_B$ is the induced permutation of $g$ on $B$). It is straightforward to verify that $\equiv$ is an equivalence relation (Exercise 2.6.2). Let $C$ be an equivalence class of $\equiv$ and $E_C = \bigcup_{B \in C} B$ (remember that the equivalence classes of $\equiv$ consist of blocks of $\mathcal{B}$), and $\mathcal{E} = \{E_C : C$ is an equivalence class of $C\}$.

**Lemma 2.6.1** Let $\Gamma$ be a digraph with $G \leq \text{Aut}(\Gamma)$ admit a normal complete block system $\mathcal{B}$ consisting of $m$ blocks of prime size $p$. Let $\equiv$ and $\mathcal{E}$ be defined as in the preceding paragraph. Then $\mathcal{E}$ is a complete block system of $G$ and for every $g \in \text{fix}_G(\mathcal{B})$, $g|_E \in \text{Aut}(\Gamma)$ for every $E \in \mathcal{E}$. Here $g|_E(x) = g(x)$ if $x \in E$ while $g(x) = x$ if $x \notin E$.

**Proof.** We will first show that $\mathcal{E}/\mathcal{B}$ is a complete block system of $G/\mathcal{B}$ by showing that $\equiv$ is a $G/\mathcal{B}$-congruence and applying Theorem 2.2.9. This will then imply that $\mathcal{E}$ is a complete block system of $G$. We thus need to show that if $B \equiv B'$ and $g \in G$, then $g(B) \equiv g(B')$ for every $g \in G$. Suppose that $g(B) \not\equiv g(B')$. Then there exists $\gamma \in \text{fix}_G(\mathcal{B})$ such that $\gamma|_{g(B)}$ is a $p$-cycle but $\gamma|_{g(B')}$ is not a $p$-cycle. Let $b \in B$. Then $g^{-1}\gamma g(b) = g^{-1}\gamma(g(b))$ and so $g^{-1}\gamma g(b) = g^{-1}\gamma g|_B$ is a $p$-cycle, while a similar argument shows that $g^{-1}\gamma g|_B$ is not. We conclude that if $B \equiv B'$ then $g(B) \equiv g(B')$ and so $\mathcal{E}$ is indeed a complete block system of $G$.

Now suppose that $B \not\equiv B'$. We will first show that in $\Gamma$, either every vertex of $B$ is our or in adjacent to every vertex of $B'$, or there is no edge between any vertex of $B$ and any vertex of $B'$. So, suppose that there is an edge from say $B$ to $B'$. As $B \not\equiv B'$, there is $\gamma \in \text{fix}_G(\mathcal{B})$ such that $\gamma|_B$ is a $p$-cycle while $\gamma|_{B'}$ is not a $p$-cycle. Raising $\gamma$ to the power $|\gamma|_{B'}$ is relatively prime to $\gamma$, we may assume without loss of generality that $|\gamma|_{B'} = 1$. Let the directed edge $b_0b' \in E(\Gamma)$, where $b_0 \in B$ and $b' \in B'$. As $\gamma|_B$ is a $p$-cycle, we may write $\gamma|_B = (b_0 b_1 \cdots b_{p-1})$ (i.e. we are writing $\gamma|_B$ as a $p$-cycle starting at $b_0$). Applying $\gamma$ to the edge $b_0b'$, we obtain the edge $b_1b'$, and applying $\gamma$ to the edge $b_0b'$ $r$ times, we obtain the edge $b_rb'$. We conclude that $bb' \in E(\Gamma)$ for every $b \in B$. Now, there exists $\delta \in \text{fix}_G(\mathcal{B})$ such that $\delta|_{B'}$ is a $p$-cycle. Applying $\delta$ to each of the edges $bb'_{p} - 1$ times (similar to above), we have that the edges $bb' \in E(\Gamma)$ for every $b \in B$ and $b' \in \mathcal{B}$. Similar arguments will show that if $b'b \in E(\Gamma)$ for some $b' \in B'$ and $b \in B$, then $bb \in E(\Gamma)$ for every $b' \in B'$ and $b \in B$, as well as if $bb' \in E(\Gamma)$ for some $b, b' \in E(\Gamma)$, then $bb' \in E(\Gamma)$ for all $b \in B$ and $b' \in B'$.

Now, let $\gamma \in \text{fix}_G(\mathcal{B})$, and consider the map $\gamma|_E$, $E \in \mathcal{E}$. If $e = x\gamma \in E(\Gamma)$ and both $x, y \in E$, then surely $\gamma|_E(e) = \gamma(e) \in E(\Gamma)$. Similarly, if both $x, y \notin E$, then $\gamma|_E(e) = e \in E(\Gamma)$. If $x \in E$ but $y \notin E(\Gamma)$, then let $B_x, B_y \in \mathcal{B}$ such that $x \in B_x$ and $y \in B_y$. Then $x\gamma' \in E(\Gamma)$ for every $x' \in B_x$, $y' \in B_y$ by arguments above. Also, $\gamma(x) = x' \in B_x$, and so $\gamma|_E(e) = x\gamma \in E(\Gamma)$. An analogous argument will show that $\gamma|_E(e) \in E(\Gamma)$ if $x \notin E$ but $y \in E$. As in every case, $\gamma|_E \in E(\Gamma)$, we have that $\gamma|_E \in \text{Aut}(\Gamma)$ establishing the result. \qed

The above result also holds in the more general situation that $\text{fix}_G(\mathcal{B})$ acts primitively on $B \in \mathcal{B}$. 
Exercise 2.6.2 Write a careful proof that $\equiv$ is an equivalence relation.

2.7 Basic Definitions Concerning Graphs

Definition 2.7.1 Let $G$ be a group and $S \subseteq G$. Define a Cayley digraph of $G$, denoted Cay$(G, S)$ to be the graph with $V(\text{Cay}(G, S)) = G$ and $E(\text{Cay}(G, S)) = \{(g, gs) : g \in G, s \in S\}$. We call $S$ the connection set of Cay$(G, S)$.

Figure 2.6: The Cayley graph Cay$(\mathbb{Z}_{10}, \{1, 3, 7, 9\})$.

If we additionally insist that $S = S^{-1} = \{s^{-1} : s \in S\}$ (or if the group is abelian and the operation is addition, that $S = -S$), then there will be no directed edges in Cay$(G, S)$, and we obtain a Cayley graph. This follows as if $(g, gs) \in E(\text{Cay}(G, S))$ and $s^{-1} \in S$, then $(gs, gs(s^{-1})) = (gs, s) \in E(\text{Cay}(G, S))$. In many situations, whether or not a Cayley digraph has loops doesn’t have any effect. In these cases the default is usually to exclude loops by also insisting that $1_G \notin S$ (or $0 \notin S$ if $G$ is abelian and the operation is addition).

Perhaps the most common Cayley digraphs that one encounters are Cayley digraphs of the cyclic groups $\mathbb{Z}_n$ of order $n$, as in Figure 2.6. A Cayley (di)graph of $\mathbb{Z}_n$ is called a circulant (di)graph of order $n$.

Definition 2.7.2 For a group $G$, the left regular representation, denoted $G_L$, is the subgroup of $S_G$ given by the left translations of $G$. More specifically, $G_L = \{x \to gx : g \in G\}$. We denote the map $x \to gx$ by $g_L$. It is straightforward to verify that $G_L$ is a group and that $G_L \cong G$.

Let $x, y \in G$, and $g = yx^{-1}$. Then $g_L(x) = yx^{-1}x = y$ so that $G_L$ is transitive on $G$.

Lemma 2.7.3 For every $S \subseteq G$, $G_L \leq \text{Aut}(\text{Cay}(G, S))$. 
PROOF. Let \( e = (g, gs) \in E(\text{Cay}(G, S)) \), where \( g \in G \) and \( s \in S \). Let \( h \in G \). We must show that \( h_1(e) \in E(\text{Cay}(G, S)) \), or that \( h_1(e) = (g', g's') \) for some \( g' \in G \) and \( s' \in S \). Setting \( g' = hg \) and \( s' = s \), we have that

\[
h_1(e) = h_1(g, gs) = (hg, h(gs)) = (hg, (hg)s) = (g', g's').
\]

\[\square\]

In general, for an abelian group \( G \), the group \( G_L \) will consist of "translations by \( g \" that map \( x \rightarrow x + g \). That is, \( G_L = \{x \rightarrow x + g : g \in G\} \). More specifically, for a cyclic group \( \mathbb{Z}_n \), we have that \( \mathbb{Z}_n \) is generated by the map \( x \rightarrow x + 1 \) (or course instead on 1, one could put any generator of \( \mathbb{Z}_n \)).

The following important result of G. Sabidussi [?] characterizes Cayley graphs.

**Theorem 2.7.4** A graph \( \Gamma \) is isomorphic to a Cayley graph of a group \( G \) if and only if \( \text{Aut}(\Gamma) \) contains a regular subgroup isomorphic to \( G \).

**Proof.** If \( \Gamma \cong \text{Cay}(G, S) \) with \( \phi : \Gamma \rightarrow \text{Cay}(G, S) \) an isomorphism, then by Lemma 2.7.3, \( \text{Aut}(\text{Cay}(G, S)) \) contains the regular subgroup \( G_L \cong G \), namely \( \phi^{-1}G_L\phi \) (see Exercise 2.7.6). Conversely, suppose that \( \text{Aut}(\Gamma) \) contains a regular subgroup \( H \cong G \), with \( \omega : H \rightarrow G \) an isomorphism. Fix \( v \in V(\Gamma) \). As \( H \) is regular, for each \( u \in V(\Gamma) \), there exists a unique \( h_u \in H \) such that \( h_u(v) = u \). Define \( \phi : V(\Gamma) \rightarrow G \) by \( \phi(u) = \omega(h_u) \). Note that as each \( h_u \) is unique, \( \phi \) is well-defined and is also a bijection as \( \omega \) is a bijection. Let \( U = \{u \in V(\Gamma) : (v, u) \in E(\Gamma)\} \). We claim that \( \phi(\Gamma) = \text{Cay}(G, \phi(U)) \).

As \( \phi(V(\Gamma)) = G \), \( V(\phi(\Gamma)) = G \). Let \( e \in E(\phi(\Gamma)) \). We must show that \( e = (g, gs) \) for some \( g \in G \) and \( s \in \phi(U) \). As \( e \in E(\phi(\Gamma)) \), \( \phi^{-1}(e) = (u_1, u_2) \in E(\Gamma) \) by Exercise 2.7.6. Let \( w \in V(\Gamma) \) such that \( h_{u_1}(w) = u_2 \). Then \( h_{u_1}^{-1}(u_1, u_2) = (v, w) \) so that \( w = h_w(v) \in U \), and \( h_{w_2} = h_{u_1}h_w \). Setting \( u_1, u_2 = (h_{u_1}(v), h_{u_2}(w)) = (h_{u_1}(v), h_{u_1}h_w(v)) = (h_{u_1}(v), h_{u_2}(v)) \).

\[
\phi(u_1, u_2) = (\phi(h_{u_1}), \phi(h_{u_2})) = (\omega(h_{u_1}), \omega(h_{u_1}h_w)) = (\omega(h_{u_1}), \omega(h_{u_1})\omega(h_w)) = (g, gs)
\]

as required. \[\square\]

We now prove a well-known result first proven by Turner [?].

**Theorem 2.7.5** Every transitive group of prime degree \( p \) contains a cyclic regular subgroup. Consequently, every vertex-transitive digraph is isomorphic to a circulant digraph of order \( p \).

**Proof.** Let \( G \) be a transitive group of prime degree \( p \). As \( G \) is transitive, it has one orbit of size \( p \), and so \( p \) divides \( |G| \). Hence \( G \) has an element of order \( p \), which is necessarily a \( p \)-cycle permuting all of the points. So \( G \) contains a regular cyclic subgroup, and the result follows by Theorem 2.7.4. \[\square\]
Exercise 2.7.6 Show that if $\phi : \Gamma \to \Gamma'$ is a graph isomorphism, then $\phi^{-1} : \Gamma' \to \Gamma$ is also a graph isomorphism. Then show that if $H \leq \text{Aut}(\Gamma)$, then $\phi^{-1}H\phi \leq \text{Aut}(\Gamma)$.

Exercise 2.7.7 Show that $(\mathbb{Z}_m)_L \wr (\mathbb{Z}_n)_L$ contains a regular subgroup isomorphic to $\mathbb{Z}_{mn}$. Consequently, the wreath product of two circulant digraphs is a circulant digraph.

Exercise 2.7.8 Show that for any two groups $G$ and $H$, $GL(n, \mathbb{Z})B$ contains a regular subgroup isomorphic to $G \times H$. Deduce that the wreath product of two Cayley digraphs is a Cayley digraph.

2.8 An Application to Graphs

Definition 2.8.1 Let $m$ and $n$ be positive integers, and $\alpha \in \mathbb{Z}_n^*$. Define $\rho, \tau : \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_m \times \mathbb{Z}_n$ by $\rho(i, j) = (i, j + 1)$ and $\tau(i, j) = (i + 1, \alpha j)$. A vertex-transitive $\Gamma$-digraph with vertex set $\mathbb{Z}_m \times \mathbb{Z}_n$ is an $(m, n)$-metacirculant digraph if and only if $\langle \rho, \tau \rangle \leq \text{Aut}(\Gamma)$.

The Petersen graph is a $(2, 5)$-metacirculant graph with $\alpha = 2$, while the Heawood graph is a $(2, 7)$-metacirculant graph with $\alpha = 6$.

Lemma 2.8.2 Let $\rho : \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_m \times \mathbb{Z}_n$ by $\rho(i, j) = (i, j + 1)$. Then $Z_{smn}(\langle \rho \rangle) = \{(i, j) \mapsto (\sigma(i), j + b_i) : \sigma \in S_n, b_i \in \mathbb{Z}_n\} = S_m t(\mathbb{Z}_n)_L$.

Proof. Straightforward computations will show that every element of $\{(i, j) \mapsto (\sigma(i), j + b_i)\}$ does indeed centralize $\langle \rho \rangle$. Then $Z_{smn}(\langle \rho \rangle)$ is transitive as $\rho \in Z_{smn}(\langle \rho \rangle)$. Additionally, $\langle \rho \rangle \triangleleft Z_{smn}(\langle \rho \rangle)$, and so the orbits $\mathcal{B}$ of $\langle \rho \rangle$ form a complete block system of $Z_{smn}(\langle \rho \rangle)$. Let $B \in \mathcal{B}$, and $g \in \text{Stab}_{Z_{smn}(\langle \rho \rangle)}(B)$. Then $g|_B$ commutes with $\langle \rho \rangle|_B$, and as $\langle \rho \rangle|_B$ is a regular cyclic group, it is self-centralizing (we have already seen that a transitive abelian group is regular in the proof of Theorem 2.2.5. The subgroup generated by any element that centralizes a regular abelian group and the regular abelian group is a transitive abelian group, and so regular.) Then $\text{Stab}_{Z_{smn}(\langle \rho \rangle)}(B)|_B \leq \langle \rho \rangle|_B$, and so by the Embedding Theorem 2.5.5, $Z_{smn}(\langle \rho \rangle) \leq S_m t(\mathbb{Z}_n)_L$. As $S_m t(\mathbb{Z}_n)_L \leq Z_{smn}(\langle \rho \rangle)$, the result follows.

Theorem 2.8.3 A vertex-transitive digraph $\Gamma$ of order $qp$, $q$ and $p$ distinct primes, is isomorphic to a $(q, p)$-metacirculant digraph if and only if $\text{Aut}(\Gamma)$ has a transitive subgroup $G$ that contains a normal complete block system $\mathcal{B}$ with $q$ blocks of size $p$.

Proof. If $\Gamma$ is isomorphic to a $(q, p)$-metacirculant, then after an appropriate relabeling, $\langle \rho, \tau \rangle \leq \text{Aut}(\Gamma)$. Then $\langle \rho \rangle \vartriangleleft \langle \rho, \tau \rangle = G$ has orbits of length $p$.

Conversely, suppose that there exists $N \triangleleft G \leq \text{Aut}(\Gamma)$ and $N$ has orbits of length $p$. Let $\mathcal{B}$ be the complete block system formed by the orbits of $N$, and assume that $G$ is the largest subgroup of $\text{Aut}(\Gamma)$ that admits $\mathcal{B}$. Then $G/\mathcal{B}$ is transitive, and so $G$ contains an element $\tau$ such that $\langle \tau \rangle/\mathcal{B}$ is cyclic of order $q$ (and so regular), and $\tau$ has order a power of $q$. By Lemma 2.6.1 there exists $\rho \in G$ such that $\langle \rho \rangle$ is semiregular of order $p$, and a Sylow $p$-subgroup $P$ of $\text{fix}_G(\mathcal{B})$ has order $p$ or $p^q$. If $|P| = p^q$, then if there is a directed edge in $\Gamma$ from some vertex of $B$ to some vertex of $B'$, $B, B' \in \mathcal{B}$, then there is a directed
exercise from every vertex of $B$ to every vertex of $B'$. We conclude that $\Gamma$ is isomorphic to a wreath product of vertex-transitive digraphs of order $q$ and $p$, respectively, and so by Theorem 2.7.5, $\Gamma$ is isomorphic to the wreath product of a circulant digraph of order $q$ and a circulant digraph of order $p$. By Exercise 2.8.4, $\Gamma$ is isomorphic to a Cayley digraph of $\mathbb{Z}_q \times \mathbb{Z}_p$, and every such digraph is isomorphic to a $(q,p)$-metacirculant digraph by Exercise 2.8.5. We henceforth assume that $|P| = p$.

Now, $(\rho)$ and $\tau^{-1}(\rho)\tau$ are contained in Sylow $p$-subgroups $P_1$ and $P_2$ of $\text{fix}_G(\mathcal{B})$, respectively, and so there exists $\delta \in G$ such that $\delta^{-1}P_2\delta = P_1$. Replacing $\tau$ with $\tau\delta$, if necessary, we assume without loss of generality that $\tau^{-1}(\rho)\tau \leq P_1$. As $|P_1| = |P_2| = p$, we see that $\tau^{-1}(\rho)\tau = (\rho)$. We now label the vertex set of $\Gamma$ with elements of $\mathbb{Z}_q \times \mathbb{Z}_p$ in such a way that $\rho(i,j) = (i, j + 1)$, and $\tau(i,j) = (i + 1, \omega_i(j))$, where $\omega_i \in S_p$, $i \in \mathbb{Z}_q$. Set $\tau^{-1}\rho\tau = \rho^a$, where $a \in \mathbb{Z}_p^*$. Define $\bar{a} : \mathbb{Z}_q \times \mathbb{Z}_p \to \mathbb{Z}_q \times \mathbb{Z}_p$ by $\bar{a}(i,j) = (i, a j)$. Then $\bar{a}^{-1}\rho^a\bar{a} = \rho$. Then $\tau\bar{a}$ centralizes $(\rho)$, and so by Lemma 2.8.2, we see that $\tau\bar{a} \in \{(i,j) \to (\sigma(i), j + b_j) : \sigma \in S_q, b_i \in \mathbb{Z}_p\}$. We conclude that $\tau\bar{a}(i,j) = (i + 1, j + b_i), b_i \in \mathbb{Z}_p$, and so $\tau(i,j) = (i + 1, a^{-1} j + c_i), c_i \in \mathbb{Z}_p$.

Let $H = \langle \tau, z_k : k \in \mathbb{Z}_q \rangle$, where $z_k(i,j) = (i, j + \delta_{ik})$, where $\delta_{ik}$ is Kronecker’s delta function. That is $\delta_{ik} = 1$ if $i = k$ and 0 otherwise. Note that $(z_k : k \in \mathbb{Z}_q) \triangleleft H$ and $H/(z_k : k \in \mathbb{Z}_q) \cong \langle \tau \rangle$. We conclude that $(\tau)$ is a Sylow $q$-subgroup of $H$. Now let, $\tau' : \mathbb{Z}_q \times \mathbb{Z}_p \to \mathbb{Z}_q \times \mathbb{Z}_p$ by $\tau'(i,j) = (i + 1, a^{-1} j)$. Then $\tau' \in H$ and also has order $|\tau|$, and so $\langle \tau \rangle$ is a Sylow $q$-subgroup of $H$ as well. Thus there exists $\gamma \in H$ such that $\gamma^{-1}(\tau)\gamma = (\tau')$. Also, $\langle \rho \rangle \triangleleft H$, and so $\gamma^{-1}(\rho)\gamma = (\rho)$, and so $\gamma^{-1}(\rho, \tau)\gamma = (\rho, \tau')$. Then $\Gamma$ is isomorphic to a $(q,p)$-metacirculant digraph.

**Exercise 2.8.4** Show that for any two groups $G$ and $H$, $G \wr H_1$ contains a regular subgroup isomorphic to $G \times H$. Deduce that the wreath product of two Cayley digraphs is a Cayley digraph.

**Exercise 2.8.5** Let $n$ be a positive integer and $n = mk$, where $\text{gcd}(m,k) = 1$. Show that any circulant digraph of order $n$ is isomorphic to an $(m,k)$-metacirculant digraph.

### 2.9 A General Strategy for Analyzing Imprimitive Permutation Groups with Blocks of Prime Size - Especially Automorphism Groups of Vertex-transitive Digraphs

Let $G$ be a transitive group that admits a complete block system $\mathcal{B}$ with blocks of prime size $p$. If $\mathcal{B}$ is not a normal complete block system, then $G/\mathcal{B}$ is a transitive faithful representation of $G$, so hopefully one can use induction... Otherwise, $\mathcal{B}$ is normal. If $\text{fix}_G(\mathcal{B})$ is not faithful on $B \in \mathcal{B}$, then in the general case, one cannot say much about $\text{fix}_G(\mathcal{B})$ other than the normalizer of a Sylow $p$-subgroup of $\text{fix}_G(\mathcal{B})$ is a vector space invariant under its normalizer, which is transitive. Tools from linear algebra may be employed - not promising but not hopeless either. In the case of the automorphism group of a vertex-transitive graph, one may employ Lemma 2.6.1 in which case the Sylow $p$-subgroup of $\text{fix}_G(\mathcal{B})$ has a very restrictive structure as we have seen. If $\text{fix}_G(\mathcal{B})$ is faithful
on $B \in \mathcal{B}$ there are three cases to consider. The first is when $\text{fix}_G(\mathcal{B}) \cong \mathbb{Z}_p$. This is the most difficult case to deal with, and nothing more will be said of this case now. If $\text{fix}_G(\mathcal{B}) \neq \mathbb{Z}_p$, then there are two subcases, depending on whether or not the action of $\text{fix}_G(\mathcal{B})$ on $B, B' \in \mathcal{B}$ are always equivalent or if they are inequivalent. We now investigate this...

**Lemma 2.9.1** Let $G \leq S_n$ be transitive on $V$ and admit a normal complete block system $\mathcal{B}$ with blocks of size $p$. Suppose that $\text{fix}_G(\mathcal{B}) \neq \mathbb{Z}_p$ is faithful on $B \in \mathcal{B}$. Define an equivalence relation $\equiv$ on $V$ by $v_1 \equiv v_2$ if and only if $\text{Stab}_{\text{fix}_G(\mathcal{B})}(v_1) = \text{Stab}_{\text{fix}_G(\mathcal{B})}(v_2)$. Then the equivalence classes of $\equiv$ are blocks of $G$, and each equivalence class of $\equiv$ contains at most one point of $B \in \mathcal{B}$.

**Proof.** As conjugation by an element of $G$ maps the stabilizer of a point in $\text{fix}_G(\mathcal{B})$ to the stabilizer of a point in $\text{fix}_G(\mathcal{B})$, $\equiv$ is a $G$-congruence, and so by Theorem 2.2.9 the equivalence classes of $\equiv$ are blocks of $G$. If a block contains two points from the same equivalence class, then by first part of this lemma applied to $\text{fix}_G(\mathcal{B})|_B$, we see that $\text{fix}_G(\mathcal{B})|_B$ is imprimitive. But a transitive group of prime degree is primitive, a contradiction. \hfill \Box

Let $\mathcal{E}$ be the complete block system consisting of the equivalence classes of $\equiv$ in the previous lemma. Suppose that each equivalence class of $\equiv$ contains exactly one element of each $B \in \mathcal{B}$. This means that $|B \cap E| = 1$ for every $B \in \mathcal{B}$ and $E \in \mathcal{E}$. Two complete $\mathcal{B}$ and $\mathcal{C}$ of $G$ such that $|B \cap C| = 1$ for every $B \in \mathcal{B}$ and $C \in \mathcal{C}$ are called **orthogonal complete block systems**. Observe that if $\mathcal{B}$ and $\mathcal{C}$ are orthogonal and $\mathcal{B}$ consists of $m$ blocks of size $k$, then $\mathcal{C}$ consists of $k$ blocks of size $m$.

**Lemma 2.9.2** Let $n = mk$ and $G \leq S_n$ such that $G$ is transitive and admits orthogonal complete block systems $\mathcal{B}$ and $\mathcal{C}$ of $m$ blocks of size $k$ and $k$ blocks of size $m$, respectively. Then $G$ is permutation equivalent to a subgroup of $S_k \times S_m$ in its natural action on $\mathbb{Z}_k \times \mathbb{Z}_m$.

**Proof.** Note that $G$ has a natural action on $\mathcal{B} \times \mathcal{C}$ given by $g(B, C) = (g(B), g(C))$, and that in this action each $g \in G$ induces a permutation contained in $S_{\mathcal{B}} \times S_{\mathcal{C}}$, namely, $(g/\mathcal{B}, g/\mathcal{C})$. Any element of $G$ in the kernel of this representation of $G$ must fix every block of $\mathcal{B}$ and every block of $\mathcal{C}$. As $|B \cap C| = 1$ for every $B \in \mathcal{B}$ and $C \in \mathcal{C}$, and there are exactly $mk = n$ such intersections, the kernel of this representation is the identity and the representation is faithful. Let $B \in \mathcal{B}$ and $C \in \mathcal{C}$. If $g \in G$ stabilizes the point $(B, C)$ in this representation, then $g(B) = B$ and $g(C) = C$. Let $B \cap C = \{x\}$. Then $g(x) = x$. Conversely, if $g(x) = x$, then there exists $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $x \in B$ and $x \in C$. Then $g(B, C) = (B, C)$ so $\text{Stab}_G(x) = \text{Stab}_G((B, C))$. It then follows by Theorem 2.3.2 that these two actions of $G$ are equivalent. \hfill \Box

Combining the two previous lemmas we have:

**Lemma 2.9.3** Let $G \leq S_n$ be transitive on $V$ and admit a normal complete block system $\mathcal{B}$ with blocks of size $p$. Suppose that $\text{fix}_G(\mathcal{B}) \neq \mathbb{Z}_p$ is faithful on $B \in \mathcal{B}$. If the action of $\text{fix}_G(\mathcal{B})$ on $B$ and $B'$ are always equivalent, then $G$ is permutation isomorphic to a subgroup of $S_{n/p} \times S_p$. 
We now illustrate these techniques by calculating the full automorphism group of circulant digraphs of order $p^2$, where $p$ is prime.

**Theorem 2.9.4** Let $\Gamma$ be a circulant digraph of order $p^2$, $p$ an odd prime. Then one of the following is true:

- $(\mathbb{Z}_{p^2})_L \leq \text{Aut}(\Gamma)$, or
- $\text{Aut}(\Gamma) = \text{Aut}(\Gamma_1) \cap \text{Aut}(\Gamma_2)$ where, $\Gamma_1$ and $\Gamma_2$ are circulant digraphs of prime order, or
- $\Gamma = \langle p^2 \rangle$ or its complement and $\text{Aut}(\Gamma) = S_{p^2}$.

Note: The result is simpler if $p = 2$, and as $|S_4| = 24$, everything can be easily determined by hand.

**Proof.** A **Burnside group** is a group $G$ with the property that whenever $H \leq S_n$ contains $G$ as a regular subgroup, then either $H$ is doubly-transitive or $H$ is imprimitive. Here, $H$ is doubly-transitive if whenever we have two order pairs $(x_1, y_1)$ and $(x_2, y_2)$ with $x_1 \neq y_1$ and $x_2 \neq y_2$, then there exists $h \in H$ such that $h(x_1, y_1) = (x_2, y_2)$. Schur showed that a cyclic group of composite order is a Burnside group [7, Theorem 3.5A]. So $\text{Aut}(\Gamma)$ is either imprimitive or doubly-transitive. If $\text{Aut}(\Gamma)$ is doubly-transitive, then $\Gamma$ is either $\langle p^2 \rangle$ or its complement and the result follows. Otherwise, $\text{Aut}(\Gamma)$ admits a complete block system $\mathcal{B}$ consisting of $p$ blocks of size $p$. (In the case $\mathbb{Z}_q \times \mathbb{Z}_p$, we still have a Burnside group, while for $\mathbb{Z}_p \times \mathbb{Z}_p$, the possibilities for a simply primitive group are given explicitly by the O'Nan–Scott Theorem.)

Let $\rho : \mathbb{Z}_{p^2} \to \mathbb{Z}_{p^2}$ by $\rho(i) = i + 1 \pmod{p^2}$, so that $\langle \rho \rangle$ is a regular subgroup of $\text{Aut}(\Gamma)$ of order $p^2$. $\mathcal{B}$ is then necessarily normal, and formed by the orbits of a normal subgroup of $\langle \rho \rangle$ of order $p$. There is a unique such subgroup, namely $\langle \rho^p \rangle$. Consider the equivalence relation $\equiv$ on $\mathcal{B}$ by $B \equiv B'$ if and only if whenever $\gamma \in \text{fix}_{c}(\mathcal{B})$ then $\gamma|_B$ is a $p$-cycle and only if $\gamma|_{B'}$ is also a $p$-cycle. By Lemma 2.6.1, the union of the equivalence classes of $\equiv$ form a complete block system $\mathcal{E}$, and $\rho^p|_E \in \text{Aut}(\Gamma)$ for every $E \in \mathcal{E}$. If $\mathcal{E}$ has blocks of size $p$, then $\mathcal{E} = \mathcal{B}$, and $\Gamma$ is isomorphic to the wreath product of two circulant digraphs of prime order. A result of Sabidussi [?] then gives (2).

If $\mathcal{E}$ consists of one block of size $p^2$, then $\text{fix}_{\text{Aut}(\Gamma)}(\mathcal{B})$ acts faithfully on $B \in \mathcal{B}$ as otherwise, as a normal subgroup of a primitive group is necessarily transitive, the kernel $K$ of the action of $\text{fix}_{\text{Aut}(\Gamma)}(\mathcal{B})$ on $B \in \mathcal{B}$ is is transitive on some $B' \in \mathcal{B}$, and so $K$ has order divisible by $p$. Then $K$ contains an element which is a $p$-cycle on $B'$ and the identity on $B$, and so $B \neq B'$. We first consider when $\text{fix}_{\text{Aut}(\Gamma)}(\mathcal{B}) \neq \mathbb{Z}_{p^2}$.

We now wish to apply a famous result of Burnside which states that a transitive group of prime degree is either permutation isomorphic to a subgroup of $\text{AGL}(1, p) = \{ x \mapsto ax + b : a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p \}$, or is a doubly-transitive group with nonabelian socle. A consequence of the Classification of the Finite Simple groups is that all doubly-transitive groups are known [3, Table], and then one can show (by examining each possible case), that a doubly-transitive group either has 1 or 2 inequivalent representations. If $H \leq \text{AGL}(1, p)$ is transitive and not isomorphic to $\mathbb{Z}_p$ (note that $p$ will divide $|H|$), then $|H| = ap$, $a > 1$. Then $\text{Stab}_H(x)$ has order $a$, and as $\text{AGL}(1, p)$ is solvable of order $p(p - 1)$, $H$ is solvable and $\gcd(a, p) = 1$. By Hall’s Theorem, any two subgroups of order $a$ are conjugate in $H$. We conclude by Theorem 2.3.2 that $H$ has a unique representation of degree $p$. 

Now define an equivalence relation $\equiv'$ on $\mathbb{Z}_{p^2}$ by $i \equiv j$ if and only if $\text{Stab}_{\text{fixAut}(\Gamma)}(i) = \text{Stab}_{\text{fixAut}(\Gamma)}(j)$. Note that as $\text{fixAut}(\Gamma)(B)$ is primitive on $B \in \mathcal{B}$, no equivalence class of $\equiv'$ can contain more than one element of $B \in \mathcal{B}$. If there is a unique representation of $\text{fixAut}(\Gamma)(B)$ as a transitive group of degree $p$, then the equivalence classes of $\equiv'$ form an orthogonal complete block system of $\text{Aut}(\Gamma)$. However, as $\mathbb{Z}_{p^2}$ contains a unique subgroup of order $p$ and every complete block system is normal, there is no such orthogonal complete block system, a contradiction! (Note that if $\mathbb{Z}_{p^2}$ is replaced with $\mathbb{Z}_p \times \mathbb{Z}_p$ of $\mathbb{Z}_q \times \mathbb{Z}_p$, there is no contradiction, but we still are done as then $\text{Aut}(\Gamma)$ is contained in a direct product and it is easy to figure out what happens). We thus assume that $\text{fix}_G(B) \cong \mathbb{Z}_p$.

Of course $\text{Aut}(\Gamma)/\mathcal{B}$ is a transitive group of prime degree, so by Burnside’s Theorem it is either contained in $\text{AGL}(1,p)$ or is a doubly-transitive group with nonabelian socle. If $\text{Aut}(\Gamma)/\mathcal{B} \leq \text{AGL}(1,p)$, then as $\text{AGL}(1,p)$ contains a normal Sylow $p$-subgroup which is necessarily $\langle \rho \rangle/\mathcal{B}$, we see that $\langle \rho \rangle \triangleleft \text{Aut}(\Gamma)$ and the result follows. I will not really talk about the other case - it doesn't really have much to do with imprimitive groups, and is also the hardest case. I will say that in the case under consideration, this cannot occur, while if $q \neq p$ it not only can occur, but in fact has two different outcomes. For $\mathbb{Z}_p \times \mathbb{Z}_p$ it also can occur but only has the obvious outcome of being something like $H \times \mathbb{Z}_p$, where $H \leq S_p$. 

2.10 Further Reading

Extensions of Burnside’s Theorem to transitive groups of degree $p^2$ as well as the full automorphism groups of all vertex-transitive digraphs of order $p^2$ can be found in [?]. An extension of Burnside’s Theorem for transitive groups that contain a regular cyclic group of prime-power order can be found in [?]. An extension of Burnside’s Theorem for groups that contain a regular abelian Hall $\pi$-subgroup is in [?]. Some information about transitive groups of degree $qp$ can be found in [?], together with the full automorphism groups of all vertex-transitive graphs of order $qp$. An extension of Burnside’s Theorem for a regular semidirect product of two cyclic groups of prime-power degree can be found in [?] (we remark that while the results in this paper are stated only for graphs, the graph structure is not used much - so the result is not explicitly stated, but can be extracted).

2.11 Selected References


Chapter 3

Leonard pairs and the q-Racah polynomials

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SUMMARY

A dual system of orthogonal polynomials arises from the Bose-Mesner algebra of an association scheme that is metric and cometric (or P- and Q-polynomial in Delsarte-Bannai's term). D. Leonard classified such orthogonal polynomials and they turned out to be in one to one correspondence with the q-Racah polynomials or certain limits of these polynomials.

P. Terwilliger interpreted the above dual system of orthogonal polynomials as two linear transformations each acting in an irreducible tridiagonal fashion on an eigenbasis of the other one. He called such linear transformations a Leonard pair. He classified them and established a representation theory for them. This immediately leads to a classification of dual systems of orthogonal polynomials as well as a characterization of the q-Racah polynomials including their limiting cases.

The theme of my lectures will be Terwilliger's theory of Leonard pairs. After a brief introduction to the relation between a Leonard pair and a dual system of orthogonal polynomials, I will introduce a raising map R and a lowering map L via the split decomposition (weight-space decomposition) attached to a Leonard pair. The eigenvalues of the Leonard pair will be written explicitly by the Askey-Wilson parameters. The key here is the tridiagonal relations (TD-relations). It turns out that the TD-relations nearly characterize Leonard pairs.

I will then introduce pre-Leonard pairs, relaxing the conditions for Leonard pairs, and ask when a pre-Leonard pair is a Leonard pair. The keys here are Terwilliger's lemma and the Askey-Wilson relations. We show that a pre-Leonard pair is a Leonard pair if and only if the data of the pre-Leonard pair, i.e., the eigenvalues together with the local traces of LR, allow Askey-Wilson parametrizations. This establishes a bijection between the set of data and the isomorphism classes of Leonard pairs. It also gives a way to construct a Leonard pair for each admissible data.

Finally I will explicitly write down dual systems of orthogonal polynomials as q-Racah polynomials and explain the limiting cases.
Leonard pairs and the $q$-Racah polynomials

— Terwilliger’s theory

1. Background in algebraic combinatorics
2. $L$-pairs and dual system of orthogonal polynomials
3. TD-pairs: weight space decomposition
4. TD-pairs: TD-relations and Terwilliger’s lemma
5. Classification of $L$-pairs
6. Classification of dual systems of orth. polynomials

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2013 Ph.D Summer School in Discrete Mathematics
Rogla, Slovenia, June 16 – June 21, 2013
1.1 Examples of 'nice' graphs

1) Johnson graph $J(v, k)$, $v \geq 2k$

$$V: \text{finite set, } |V| = v$$

$$X = \binom{V}{k}: \text{the set of } k\text{-subsets of } V$$

$$R \ni (x, y) \iff |x \cap y| = k - 1$$

$$d = k \text{ (diameter)}$$

2) $\varphi$-Johnson graph $J_\varphi(v, k)$, $v \geq 2\varphi$

$$V: \text{v-dim vector space } / \mathbb{F}_q$$

$$X = \binom{V}{k}: \text{the set of } k\text{-dim subspaces of } V$$

$$R \ni (x, y) \iff \dim x \cap y = k - 1$$

$$d = k \text{ (diameter)}$$
(3) Hamming graph \( H(n, q) \)

- \( F \) : finite set, \( |F| = q \)
- \( X = F^n = F \times F \times \cdots \times F \) \( \Rightarrow x = (x_1, x_2, \ldots, x_n) \)
- \( y = (y_1, y_2, \ldots, y_n) \)
- \( R \circ (x, y) \iff \# \{ i \mid x_i \neq y_i \} = 1 \)
- \( d = n \) (diameter)

(4) Bilinear forms graph \( \text{Bil}(m, q) \)

- \( X \) : the set of \( m \times n \) matrices over \( \mathbb{F}_q \)
- \( R \circ (x, y) \iff \text{rank}(x-y) = 1 \)
- \( d = \text{Min} \{ m, n \} \) (diameter)

How nice?

\( \Gamma = (X, R) \) : finite, connected, simple graph with diameter \( d \)

- \( A \) : the adjacency matrix of \( \Gamma \)
  \[
  A(x, y) = \begin{cases} 
  1 & \text{if } (x, y) \in R \\
  0 & \text{otherwise}
  \end{cases}
  \]

- \( A_i \) : \( i \)-th distance matrix
  \[
  A_i(x, y) = \begin{cases} 
  1 & \text{if } d(x, y) = i \\
  0 & \text{otherwise}
  \end{cases}
  \]

In particular,

\( A = A_1 \)

1st nice property:

\[ \exists v_i(x) \in \mathbb{Q}[x] \text{ polynomial of degree } i \]

s.t.

\[ A_i = v_i(A) \]

Distance-regular, \( P \)-polynomial
\[ V = \mathbb{R}^X = \left\{ f : X \to \mathbb{R} \right\} \cong \mathbb{R}^n, \quad |X| = n \]

\[ M_X(\mathbb{R}) = \left\{ B : X \times X \to \mathbb{R} \right\} \cong M_n(\mathbb{R}) \]

\[ M_X(\mathbb{R}) \text{ acts on } V = \mathbb{R}^X \]

\[ (B f)(x) = \sum_{y \in X} B(x, y) f(y) \]

\[ M_X(\mathbb{R}) \supset \Omega = \langle A_0, A_1, A_2, \ldots, A_d \rangle \]

\[ \mathbb{I}, \mathbb{A} \quad \text{Bose-Hamner algebra} \]

\[ \dim \Omega = d + 1 \]

\[ V = V_0 + V_1 + \ldots + V_d \quad \text{eigenvalue decomposition of } A \]

\[ \lambda_i : \text{eigenvalue of } A \text{ on } V_i \]

\[ E_i : V \rightarrow V_i \quad \text{projection} \]

\[ A = \lambda_0 E_0 + \lambda_1 E_1 + \ldots + \lambda_d E_d \]

\[ M_X(\mathbb{R}) \supset \Omega = \langle A_0, A_1, \ldots, A_d \rangle \]

\[ = \langle E_0, E_1, \ldots, E_d \rangle \]

\[ \begin{cases} I & = E_0 + E_1 + \ldots + E_d \quad \text{primitive idempotents} \\ E_i E_j & = \delta_{ij} E_i \end{cases} \]

\[ M_X(\mathbb{R})^\circ : \text{the algebra } M_X(\mathbb{R}) \text{ w.r.t.} \]

\[ \text{the entry-wise product } \circ \quad \text{(Hadamard product, Schur product)} \]

\[ (B_1 \circ B_2)(x, y) = B_1(x, y) B_2(x, y) \]

\[ M_X(\mathbb{R})^\circ \supset \Omega^\circ = \langle E_0, E_1, \ldots, E_d \rangle \quad \text{dual BM-alg.} \]

\[ = \langle A_0, A_1, \ldots, A_d \rangle \]

\[ \begin{cases} J & = A_0 + A_1 + \ldots + A_d \\ \text{all one matrix (the identity of } M_X(\mathbb{R})^\circ) \\ A_i \circ A_j & = \delta_{ij} A_i \end{cases} \]

\[ 2\text{nd new property} \]

\[ \exists v_i^*(x) \in \mathbb{R}[x] \quad \text{polynomial of degree } i \]

\[ \text{st.} \quad n F_i = v_i^*(nE_i), \quad n = |X| \]

\[ \mathbb{Q}-\text{polynomial} \quad \text{in } \Omega^\circ \]
P- and Q-polynomial scheme

\[ \Gamma(X, R) : \text{finite, connected simple graph} \]
\[ \text{diameter } d, \quad |X| = n \]

\[ M_X(R) \supset \mathcal{G} = \langle A_0, A_1, \ldots, A_d \rangle \quad \text{B.M.-algebra} \]
\[ \mathcal{G} = \langle A_0, E_0, \ldots, E_d \rangle \quad \text{primitive idempotents} \]
\[ A = A_1 = \otimes_0 E_0 + \otimes_1 E_1 + \ldots + \otimes_d E_d \]
\[ M_X(R^G) \supset \mathcal{G}^G = \langle nE_0, nE_1, \ldots, nE_d \rangle \quad \text{dual B.M.-algebra} \]
\[ \mathcal{G}^G = \langle A_0, A_1, \ldots, A_d \rangle \]
\[ nE = nE_1 = \otimes_0^* A_0 + \otimes_1^* A_1 + \ldots + \otimes_d^* A_d \]

\[ \exists \nu_i(x), \nu_i^*(x) \in R[x] \quad \text{polynomials of degree } i \quad (0 \leq i \leq d) \]
\[ \mathcal{A}_i = \nu_i(A) \]
\[ n\mathcal{E}_i = \nu_i^*(nE) \]

Fact

\[ \Gamma(X, R) : \text{P- and Q-polynomial} \]

\[ \Rightarrow \]

\[ \{ \nu_i(x) \}_{i=0}^d : \text{orthogonal polynomial} \]
with support \{\theta_0, \theta_1, \ldots, \theta_d\}

\[ \{ \nu_i^*(x) \}_{i=0}^d : \text{orthogonal polynomial} \]
with support \{\theta_0^*, \theta_1^*, \ldots, \theta_d^*\}

\[ \frac{\nu_i(\theta_j)}{\nu_i(\theta_0)} = \frac{\nu_i^*(\theta_j)}{\nu_i^*(\theta_0^*)} \quad \text{all } i, j \]

\[ \text{dual system of orthogonal polynomials} \]
Leonard Theorem

\[ \{ v_i^\alpha \}_{\alpha=0}^d, \{ v_i^\beta \}_{\beta=0}^d : \text{dual system of orth. proj.} \]

\[ \Rightarrow \]

\[ \left\{ \frac{1}{v_i(x)} v_i^\alpha(x) \right\}_{\alpha=0}^d, \left\{ \frac{1}{v_i(x)} v_i^\beta(x) \right\}_{\beta=0}^d \]

see q-Racah polynomials
(Ashey - Wilson polynomials)

Then limits

Leonard 1982
Bannai - Ito 1984
Terwilliger 2001

in terms of Leonard pairs

\[ \Gamma = (X, R) \] finite connected simple graph

P- and Q-polynomial

\[ X \ni x_0 \] base point fixed

\[ X_i = X_i(x_0) = \{ x \in X \mid \deg(x_0 x) = i \} \]

\[ V_i^\ast = V_i(x_0) = \{ f : X \to \mathbb{C} \mid f(x_0) = 0 \land f(x) \neq X_i \} \]

Then

\[ V = \bigoplus \limits_{i=0}^d V_i^\ast \]

\[ E_i^\ast = E_i^\ast(x_0) : V \longrightarrow V_i^\ast \] projection

\[ \mathcal{O}^\ast = \mathcal{O}^\ast(x_0) = \langle E_0^\ast, E_1^\ast, \ldots, E_d^\ast \rangle \subset M_X^\ast \mathbb{C} \cong \text{End}(V) \]

\[ \left\{ I = E_0^\ast + E_1^\ast + \ldots + E_d^\ast \right\} \]

\[ E_i E_j = \delta_{ij} E_i^\ast \] primitive idempotents

Terwilliger algebra
\[\sigma^0 = \{E_0, E_1, \ldots, E_d\} \subset M_X(C)^0, \quad \text{dual BM-alg} \]
\[\sigma^* = \{E_0, E_1, \ldots, E_d\} \subset M_X(C), \quad \text{BM-alg} \]

\[
\begin{align*}
\text{BM-alg} & \quad \text{dual BM-alg} \\
\sigma^0 & \quad \sigma^* \\
\text{dual BM-alg} & \quad \text{BM-alg}
\end{align*}
\]

\[n\ E = n\ E_i = \sigma_i^* A_0 + \sigma_i^* A_1 + \cdots + \sigma_i^* A_d \]

\[M_X(C) \quad M_X(C)^0 \]

\[E_i : V = \bigoplus_{j=0}^d V_j \longrightarrow V_i \quad \text{projection} \]

\[E_i^* : V = \bigoplus_{j=0}^d V_j^* \longrightarrow V_i^* \quad \text{projection} \]

**Proposition**

\[A V_i^* \leq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \leq i \leq d \]

\[A^* V_i \leq V_{i-1} + V_i + V_{i+1}, \quad 0 \leq i \leq d \]

More precisely

\[E_j^* A E_i^* = 0 \quad \text{if} \quad |i-j| > 1 \]
\[E_j^* A E_i^* \neq 0 \quad \text{if} \quad |i-j| = 1 \]

\[E_j^* A^* E_i = 0 \quad \text{if} \quad |i-j| > 1 \]
\[E_j^* A^* E_i \neq 0 \quad \text{if} \quad |i-j| = 1 \]
Proposition

\[ T = T(T) \text{ is a semi-simple algebra.} \]

In particular
\[ V = \mathbb{C}^X \text{ is a direct sum of irreducible} \]
\[ T \text{-submodules} \]

Corollary
\[ V = \mathbb{C}^X \supset W : \text{irreducible} \]
\[ T \text{-submodule} \]
\[ \Rightarrow \]
\[ A|_W, A^*|_W \in \text{End}(W) \]
\[ \text{and a TD-pair.} \]

\[ \text{TD-pair (tridiagonal pair)} \]

\[ W : \text{finite dim. vector space over} \mathbb{C} \]
\[ A, A^* \in \text{End}(W) : \text{diagonalizable} \]
\[ W = \bigoplus_{i=0}^{d} W_i \quad \text{eigenspace decom. of} \ A \]
\[ W = \bigoplus_{i=0}^{d^*} W_i^* \quad \text{eigenspace decom. of} \ A^* \]

The pair \( A, A^* \) is called a TD-pair if

1. \( A W_i^* \subseteq W_{i-1}^* + W_i^* + W_{i+1}^* \), \( 0 \leq i \leq d^* \)
   \( (W_i^* = W_{d^*+1}^* = 0) \)

2. \( A^* W_i \subseteq W_{i-1} + W_i + W_{i+1} \), \( 0 \leq i \leq d \)
   \( (W_{d-1} = W_{d+1} = 0) \)

and

4. \( W \) is irreducible as an \( \langle A, A^* \rangle \)-module.

Remark \( d = d^* \)
Def

\[ A, A^* \in \text{End}(W) : \quad L \text{-pair } (\text{Leonard pair}) \]

\[ \iff \]

- T-pair

and

\[ \dim V_i = \dim V_i^* = 1, \quad 0 \leq i \leq d. \]

Example

\[ T = T(x_0) = \langle A, A^* \rangle \subset \text{End}(V), \quad V = \mathbb{C}^X \]

Tarwell's alg. \text{of a } \Gamma \text{-and } Q\text{-poly scheme}

Set

\[ W = \{ f \in V \mid f \text{ is constant on each } X_i \}, \]

where

\[ X_i = \{ x \in X \mid g(x, x) = i \}. \]

Then \( W \) is an \text{indecisile } \( T \)-module

and

\[ \dim W \cap V_i = 1, \quad 0 \leq i \leq d \]

\[ \dim W \cap V_i^* = 1, \quad 0 \leq i \leq d. \]

\( W \) is called the \text{principal } \( T \)-module

(\text{primary } \( T \)-module)

\section{2.1 \text{Orthogonal polynomials} \( \mathbb{C} \\text{ with finite support} \)

\[ u_0(x) = 1, \quad u_1(x), \quad \ldots, \quad u_d(x) \in \mathbb{C}[x] \]

\[ \deg u_i(x) = i, \quad 0 \leq i \leq d \]

\text{Orthogonal polynomials} \( \text{with support } \beta_0, \beta_1, \ldots, \beta_d \in \mathbb{C} \)

\[ (\beta_i \neq \beta_j ) \]

\[ \sum_{\nu=0}^d u_\nu(\beta_i) u_\nu(\beta_j) \mu_\nu = \frac{1}{k_i} \delta_{ij}, \quad \text{all } i, j \]

for some \( \mu_0, \mu_1, \ldots, \mu_d \in \mathbb{C} \)

\[ \mu_0 \neq 0, \quad 0 \leq i \leq d, \quad \sum_{\nu=0}^d \mu_\nu = 1 \]

and \( k_0, k_1, \ldots, k_d \in \mathbb{C} \)

\[ k_k \neq 0, \quad 0 \leq i \leq d, \quad k_0 = 1. \]
\[
\{ u_i(x) \}_{i=0}^d \quad \text{or k, poly, with support } \{ \theta_i \}_{i=0}^d \\
\]

\[
(\ast ) \quad x u_i(x) = c_i u_{i-1}(x) + a_i u_i(x) + b_i u_{i+1}(x), \quad 0 \leq i \leq d \\
\text{where } u_0(x) = 0, \quad b_d = 1 \\
u_{d+1}(x) = \frac{1}{b_0 b_1 \ldots b_{d-1}} (x-\theta_0)(x-\theta_1) \ldots (x-\theta_d) \\
B = \begin{pmatrix}
q_0 & b_0 & 0 & & \\
c_1 & a_1 & b_1 & & \\
& \ddots & \ddots & \ddots & \\
0 & c_{d-1} & a_{d-1} & b_{d-1} & \\
& & 0 & c_d & a_d
\end{pmatrix} \quad \text{triangular matrix}
\]

\[
\text{tridiagonal, indiagonal, with eigenvalues } \{ \lambda_i \}_{i=0}^d \\
\Rightarrow
\]

Conversely

\[
B: \quad \text{tridiagonal, indiagonalize with eigenvalues } \{ \theta_i \}_{i=0}^d
\]

\[
\Rightarrow
\]

\[
\{ u_i(x) \}_{i=0}^d \quad \text{defined by } (\ast ) \\
\text{or k, poly, with support } \{ \theta_i \}_{i=0}^d \\
\]

\[
k_0 = 1, \quad k_i = \frac{b_0 b_1 \ldots b_{i-1}}{c_1 c_2 \ldots c_i}, \quad 0 \leq i \leq d \\
\mu_i = \frac{1}{k_d u_{d+1}(\theta_i) u_d(\theta_i)} \quad \text{Christoffel number}
\]

Remark

\[
B: \quad \text{tridiagonal, indiagonal, diagonalizable}
\]

\[
\Rightarrow
\]

Each eigenvalue of B has multiplicity 1.
$\mathcal{P} = \{ \text{orth. polys. with finite support} \}$

$\mathcal{M} = \{ \text{tridiagonal matrices that are irreducible and diagonalizable} \}$

$\mathcal{P} \overset{1:1}{\longrightarrow} \mathcal{M}$

$\mathcal{P} \subset \mathcal{P}(\theta_0) = \{ \{ u_i(x) \} \in \mathcal{P} \mid u_i(\theta_0) = 1 \ \text{all } i \}$

$\mathcal{M} \subset \mathcal{M}(\theta_0) = \{ B \in \mathcal{M} \mid \text{row sum of } B = \theta_0 \}$

$\mathcal{P}(\theta_0) \overset{1:1}{\longrightarrow} \mathcal{M}(\theta_0)$

§2.2 dual systems of orth. polys. and L-pairs

\[
\begin{align*}
\{ u_i(x) \}_{i=0}^d & \text{ with polys. with support } \{ \theta_v \}_{v=0}^d \\
\{ u_i^*(x) \}_{i=0}^d & \text{ with polys. with support } \{ \theta_v^* \}_{v=0}^d
\end{align*}
\]

dual ordering $\theta_0, \theta_1, \ldots, \theta_d$

ordering $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$

\[ u_i(\theta_j) = u_i^*(\theta_j^*) \quad \text{all } i, j \]

Remark

\[ u_j(\theta_0) = u_j^*(\theta_0^*) = 1 \quad \text{all } j \]

\[ u_i^*(\theta_i^*) = u_i(\theta_i) = 1 \quad \text{all } i \]

\[ \{ u_i(x) \}_{i=0}^d \in \mathcal{P}(\theta_0), \quad \{ u_i^*(x) \}_{i=0}^d \in \mathcal{P}(\theta_0^*) \]
\[ P(\Theta) \Rightarrow \{ u_i(x) \}_{i=0}^{d} \quad \Longleftrightarrow \quad B \in M(\Theta) \]

\[ P(\Theta^*) \Rightarrow \{ u_i^*(x) \}_{i=0}^{d} \quad \Longleftrightarrow \quad B^* \in M(\Theta^*) \]

**Proposition**

\[ \{ u_i(x) \}_{i=0}^{d}, \{ u_i^*(x) \}_{i=0}^{d} \text{ are dual} \]

\[ \iff \exists S \text{ non singular matrix} \]

\[ SBS = \begin{pmatrix} \Theta_0 & 0 \\ 0 & \Theta_d \end{pmatrix} = D \text{ diagonal matrix} \]

\[ SBS^{-1} = \begin{pmatrix} \Theta_0^* & 0 \\ 0 & \Theta_d^* \end{pmatrix} = D^* \text{ diagonal matrix} \]

\[ B, D^* \in M_{d+1}(\Theta) \cong \text{End}(V), \quad V = \mathbb{C}^{d+1} \]

**Eigenspace decomposition of** $B$

\[ V = V_0 \oplus V_1 \oplus \cdots \oplus V_d, \quad \text{dim } V_i = 1, \quad 0 \leq i \leq d \]

\[ V_i = \langle e_i \rangle, \quad B e_i = \lambda_i e_i \]

**Eigenspace decomposition of** $D^*$

\[ V = V_0^* \oplus V_1^* \oplus \cdots \oplus V_d^*, \quad \text{dim } V_i^* = 1, \quad 0 \leq i \leq d \]

\[ V_i^* = \langle e_i^* \rangle, \quad D^* e_i = \lambda_i^* e_i \]

(i) \[ B V_i = V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \leq i \leq d, \quad V_i = V_{d+1}^* = 0 \]

$B$: tri-diagonal

\[ D^* V_i = V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \leq i \leq d, \quad V_i = V_{d+1}^* = 0 \]

$D^*$: tri-diagonal

(ii) \[ V: \text{inducible as a } \langle B, D^* \rangle \text{-module} \]

$B$: inducible tri-diagonal
\( A, A^* \in \text{End}(V) \) diagonalizable, \( V \cong \mathbb{C}^{d+1} \)

\[
V = \bigoplus_{i=0}^{d} V_i, \quad \text{dim} \ V_i = 1, \quad 0 \leq i \leq d
\]

\[ A \mid V_i = \lambda_i \in \mathbb{C}, \quad \lambda_i \neq \lambda_j \]

eigenspace decomposition of \( A \)

\[
V = \bigoplus_{i=0}^{d} V_i^*, \quad \text{dim} \ V_i^* = 1, \quad 0 \leq i \leq d
\]

\[ A^* \mid V_i^* = \lambda_i^* \in \mathbb{C}, \quad \lambda_i^* \neq \lambda_j^* \]

eigenspace decomposition of \( A^* \)

\[ L \text{- pair (Leonard pair)} \]

if

\[ \exists \text{ ordering } \lambda_0, \lambda_1, \ldots, \lambda_d \]

\[ \exists \text{ ordering } \lambda_0^*, \lambda_1^*, \ldots, \lambda_d^* \]

\[
\begin{cases}
AV_i^* \leq V_{i-1}^* + V_i^* + V_{i+1}^*, & 0 \leq i \leq d, \quad V_{i-1} = V_i = 0 \\
A^*V_i \leq V_{i-1} + V_i + V_{i+1}, & 0 \leq i \leq d, \quad V_{i-1} = V_i = 0
\end{cases}
\]

and

\( V \) is indecomposable as an \( \langle A, A^* \rangle \)-module.
\[ \mathcal{P}(I_{n}) \ni \{ u_{i}^{(x)} \}_{i=0}^{d} \iff B \in M(I_{n}) \]

\[ \mathcal{D}(I_{n}) \ni \{ u_{i}^{(x)} \}_{i=0}^{d} \iff B^{*} \in M(I_{n}^{*}) \]

\[ \{ u_{i}^{(x)} \}_{i=0}^{d}, \{ u_{i}^{(x)} \}_{i=0}^{d} : \text{dual} \]

\[ \iff B, B^{*} : \text{L-pair} \]

In this case,

\[ (B, B^{*}) \cong (D, D^{*}) \quad \text{isomorphic as L-pairs} \]

\[ \text{Remark} \]

\[ A, A^{*} \in \text{End}(V) \quad \text{L-pair} \]

Then, \exists \text{ a basis of } V, \text{ the matrix } B \text{ of } A \in M(I_{n})

the matrix \( D^{*} \) of \( A^{*} \) is diagonal

and

\[ \exists \text{ a basis of } V, \text{ the matrix } D \text{ of } A \text{ is diagonal} \]

the matrix \( B^{*} \) of \( A^{*} \) \in M(I_{n}^{*})

\[ \text{§2.3 Leonard's theorem - classification of dual systems of orth. poly.} \]

\[ (a_{i})_{i} = a(a+1) \cdots (a+i-1), \quad i = 1, 2, \ldots \]

\[ (a_{0})_{i} = 1 \quad \text{shifted factorial} \]

\[ \phi_{r}^{(a_{1}, \ldots, a_{n}; b_{1}, \ldots, b_{n}; q)}(x) = \sum_{i=0}^{\infty} \frac{(a_{1})_{i} \cdots (a_{n})_{i} \cdots (b_{1})_{i} \cdots (b_{n})_{i}}{i!} \]

\[ \text{hypogeometric series} \]

\[ (a_{i} q_{i}) = (1-a) \cdots (1-a q_{i}^{1-i}), \quad i = 1, 2, \ldots \]

\[ (a_{0} q_{0}) = 1 \quad \text{q-shifted factorial} \]

\[ \phi_{r}^{(a_{1}, \ldots, a_{n}; b_{1}, \ldots, b_{n}; q)}(x) = \sum_{i=0}^{\infty} \frac{(a_{1} q_{1})_{i} \cdots (a_{n} q_{n})_{i} \cdots (b_{1})_{i} \cdots (b_{n})_{i}}{(q q_{i})_{i}} \]

\[ \text{basic hypegeometric series} \]

\[ (q \text{- hypegeometric series}) \]

\[ \phi \to 1, \quad \phi \to \phi^{r} \]
Theorem (Leonard 1982, Bannai-Ito 1984)

\( \{ u_i(x) \}_{i=0}^{d}, \{ u_i^*(x) \}_{i=0}^{d} \) : a dual system of orthogonal polynomials with support \( \{ \alpha_i^{(u)} \}, \{ \alpha_i^{(q)} \} \) respectively

\[ u_i(x) = \Phi_i \left( \frac{q^{x+i}}{q^i}, \frac{s^{q^{x+i}}}{q^i}, \frac{q^{x+i}}{q^i}, \frac{s^{q^{x+i}}}{q^i} \right) \]

\[ x = \phi_i(y) = \phi_i + \kappa \frac{1}{q^y} (1 - q^y)(1 - s^{q^{y+i}}) \]

\[ \phi_i = \phi_i(y), \quad 0 \leq i \leq d \]

\[ u_i^*(x) = \Phi_i \left( \frac{q^{x+i}}{q^i}, \frac{s^{q^{x+i}}}{q^i}, \frac{q^{x+i}}{q^i}, \frac{s^{q^{x+i}}}{q^i} \right) \]

\[ x = \phi_i(y) = \phi_i + \kappa \frac{1}{q^y} (1 - q^y)(1 - s^{q^{y+i}}) \]

\[ \phi_i^* = \phi_i(y), \quad 0 \leq i \leq d \]

(details in 6.4 lecture)

\[ V = \bigoplus_{i=0}^{d} u_i \quad \dim U_i = 1, \quad 0 \leq i \leq d \]

Weight space decomposition

(split decomposition)
\[ F_i : V \rightarrow U_i \text{ projection} \]

\[ R = A - \sum_{i=0}^{d} \lambda_i F_i, \quad \lambda_i : \text{eigenvalue of } A \text{ on } V_i \]

\[ L = A^* - \sum_{i=0}^{d} \lambda_i^* F_i, \quad \lambda_i^* : \text{eigenvalue of } A^* \text{ on } V_i^* \]

Then

\[
\begin{aligned}
R U_i &= U_{i+1}, \quad 0 \leq i \leq d, \quad U_{d+1} = 0 \\
L U_i &= U_{i-1}, \quad 0 \leq i \leq d, \quad U_{-1} = 0 
\end{aligned}
\]

\[ \begin{array}{c}
\text{R raises} \{ U_i \}_{i=0}^{d} \\
\text{L lowers} \{ U_i \}_{i=0}^{d}
\end{array} \]

Conversely, start with

\[ V = \bigoplus U_i, \quad \dim U_i = 1 \]

\[ F_i : V \rightarrow U_i \text{ projection} \]

Given

\[
\begin{aligned}
&\theta_0, \theta_1, \ldots, \theta_d \in \mathbb{C}, \quad \theta_i \neq \theta_j; \\
&\theta_0^*, \theta_1^*, \ldots, \theta_d^* \in \mathbb{C}, \quad \theta_i^* \neq \theta_j^*
\end{aligned}
\]

and

\[ R, L \in \text{End}(V) \text{ raising, lowering maps} \]

\[
\begin{aligned}
\text{R} & U_i = U_{i+1}, \quad 0 \leq i \leq d, \quad U_{d+1} = 0 \\
\text{L} & U_i = U_{i-1}, \quad 0 \leq i \leq d, \quad U_{-1} = 0
\end{aligned}
\]

Set

\[
\begin{aligned}
A &= R + \sum_{i=0}^{d} \lambda_i F_i \\
A^* &= L + \sum_{i=0}^{d} \lambda_i^* F_i
\end{aligned}
\]

Such \( A, A^* \in \text{End}(V) \) are called a \textit{pre-L-pair}. 
Choose \( u_0 \in U_0 \), \( u_0 \neq 0 \).

Set \( u_i = R^{x_i} u_0 \).

Then \( U_i = C u_i \), \( 0 \leq i \leq d \)

and \( L u_{i-1} = \lambda_i u_i \), \( 0 \leq i \leq d-1 \)

for some \( \lambda_i \in C^\times = C - \{0\} \).

The data \( \left( \{\alpha_i\}_{i=0}^d, \{\alpha_i^*\}_{i=0}^d, \{\lambda_i\}_{i=0}^{d-1} \right) \)

determines the isomorphism class of a pre L-pair \( A, A^* \in \text{End}(V) \).

If \( \alpha_i \neq 0 \), \( \alpha_i^* \neq 0 \), \( i \neq j \in \{0, 1, \ldots, d\} \)

and \( \lambda_i \neq 0 \), \( 0 \leq i \leq d-1 \),

then \( \left( \{\alpha_i\}_{i=0}^d, \{\alpha_i^*\}_{i=0}^d, \{\lambda_i\}_{i=0}^{d-1} \right) \)

is the data of some pre L-pair \( A, A^* \).

**Theorem (Templer 2001)**

\( A, A^* \in \text{End}(V) \) is an L-pair

with data \( \left( \{\alpha_i\}_{i=0}^d, \{\alpha_i^*\}_{i=0}^d, \{\lambda_i\}_{i=0}^{d-1} \right) \).

Then \( A, A^* \) are an L-pair

\[ \Leftrightarrow \]

\( \alpha_i = \beta_0 + 2 \frac{1}{q^i} (1 - q^i)(1 - q^{d+1}) \), \( 0 \leq i \leq d \)

\( \alpha_i^* = \beta_0^* + 2 \frac{1}{q^i} (1 - q^i)(1 - q^{d+1}) \), \( 0 \leq i \leq d \)

\( \lambda_i = k \frac{q^{2i+1}}{q^{d+1}} (1 - q^{d+1})(1 - q^{d-1})(1 - q^{d+1})(1 - q^{d+1}) \), \( 0 \leq i \leq d \)

for some \( \gamma, \gamma_t, s, s^t, \beta_0, \beta_0^* \), \( k \in C \)

\( r_i, r_t = s^t q^{d+1} \)

\( \beta = k \neq 0 \), \( q^t \neq 1 \), \( 1 \leq i \leq d \)

\( s, s^t \neq \{q^t, q^{d-1}, \ldots, q^{d+1}\} \)

\( r_t, r_t \neq \{q^t, q^{d-1}, \ldots, q^{d+1}\} \cup \{s_t, s^{t+1}, \ldots, s^{t+1}\} \) if \( s^t \neq 0 \)

\( r_t = 0 \), \( r_t \neq \{q^t, q^{d-1}, \ldots, q^{d+1}\} \cup \{s_t, s^{t+1}, \ldots, s^{t+1}\} \) if \( s^t = 0 \)

or the limiting case

(details in 5th lecture)
§3 TD-pairs: weight space decomposition

$V$: finite dim vector space /C

$A, A^* \in \text{End}(V)$ diagonalizable

$V = \bigoplus_{i=0}^{d} V_i$ eigenspace decomp. of $A$

$V = \bigoplus_{i=0}^{d} V_i^*$ eigenspace decomp. of $A^*$

$A, A^*$: TD-pair (tridiagonal pair)

(i) $\exists$ standard ordering $V_0, V_1, ..., V_d$

(ii) $\exists$ standard ordering $V_i^*, V_i^{*+}, ..., V_d^*$

$AV_i \subseteq V_i + V_{i+1}^*, 0 \leq i \leq d^*$, $V_d^* = V_{d+1}^* = 0$

$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}, 0 \leq i \leq d$, $V_i = V_{d+1} = 0$

and

(ii) $V$ is irreducible as an $(A, A^*)$-module

$V \nmid W$ $A$-inv, $A^*$-inv. subspace

$\Rightarrow W = V$ or $0$.

Remark

1. $d = d^*$ holds. diameter

   trivial TD-pair if $d=0$.

   We assume $d \geq 1$ unless otherwise stated.

2. $A, A^* \in \text{End}(V)$: L-pair (Leonard pair)

   If $\dim V_i = \dim V_i^* = 0$, $0 \leq i \leq d$.

3. $V_0, V_1, ..., V_d$: standard

   $\Rightarrow V_0, V_{d+1}, ..., V_{d}$: standard

   and no other standard orderings

   for the eigenspaces of $A$.

   The same holds for the eigenspaces of $A^*$.

   $(A, A^*; \{V_i\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d})$ TD-system

   3 more TD-systems

   $(\{V_{d-i}\}_{i=0}^{d}, \{V_{d-i}^*\}_{i=0}^{d})$

   $(\{V_i\}_{i=0}^{d}, \{V_{d-i}^*\}_{i=0}^{d})$

   $(\{V_i\}_{i=0}^{d}, \{V_{d-i}^*\}_{i=0}^{d})$
In what follows, we fix a TD-system

\[(A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)\].

Set

\[U_i = (V_i^* + \cdots + V_d^*) \cap (V_i + \cdots + V_d), \quad 0 \leq i \leq d\]

weight space

\[U_0 = V_0^*, \quad U_d = V_d\]

\[\lambda_i : \text{eigenvalue of } A \text{ on } V_i\]
\[\lambda_i^* : \text{eigenvalue of } A^* \text{ on } V_i^*\]

Lemma

\[(A - \lambda_i) U_i \subseteq U_{i+1}, \quad 0 \leq i \leq d, \quad U_{d+1} = 0\]

\[(A^* - \lambda_i^*) U_i \subseteq U_{i-1}, \quad 0 \leq i \leq d, \quad U_{-1} = 0\]

Proposition

(1) \[V = \bigoplus_{i=0}^d U_i\]

(weight space decomposition)

\[\begin{align*}
U_0 + \cdots + U_i &= V_0^* + \cdots + V_i^*, \quad 0 \leq i \leq d \\
U_i + \cdots + U_d &= V_i + \cdots + V_d, \quad 0 \leq i \leq d
\end{align*}\]

(3) \[\dim U_i = \dim V_i = \dim V_i^*, \quad 0 \leq i \leq d\]

(4) \[\dim U_i = \dim U_{d-i}, \quad 0 \leq i \leq d\]
Proof

(1) \[ W = U_0 + U_1 + \ldots + U_d \quad \text{is} \quad \langle A, A^\ast \rangle - \text{inv.} \]
\[ \geq U_0 = V_0^\ast \neq 0 \quad \text{by Lemma} \]
\[ S_0, \quad W = V \quad \text{by the irreducibility of } V \]
\[ (U_0 + \ldots + U_1) \cap (U_{i+1} + \ldots + U_d) \]
\[ \leq (V_0^\ast + \ldots + V_i^\ast) \cap (V_{i+1} + \ldots + V_d) \]
\[ =: U_i^\ast \]
\[ W' = U_0^\ast + U_1^\ast + \ldots + U_d^\ast \quad \text{is} \quad \langle A, A^\ast \rangle - \text{inv.} \]
\[ \leq V_0^\ast + \ldots + V_d^\ast \neq V \]
\[ (A - \theta_{i+1}) U_i^\ast \subseteq U_i^\ast, \quad \quad (A - \theta_2) U_d^\ast = 0 \]
\[ (A^\ast - \theta_i^\ast) U_i^\ast \subseteq U_i^\ast, \quad \quad (A^\ast - \theta_0^\ast) U_0^\ast = 0 \]

So \[ W' = 0 \]
\[ (U_0 + \ldots + U_i) \cap (U_{i+1} + \ldots + U_d) = 0 \]

(2) \[ U_0 + \ldots + U_1 \subseteq V_0^\ast + \ldots + V_i^\ast \]
\[ V_0^\ast + \ldots + V_i^\ast = (A^\ast - \theta_{i+1}) \ldots (A^\ast - \theta_2) V \]
\[ \leq U_0 + \ldots + U_i \quad \text{by Lemma} \]
\[ + V = \bigoplus_{i=0}^d U_i \]
\[ U_0 + \ldots + U_d \subseteq V_0 + \ldots + V_d \]
\[ V_0 + \ldots + V_d = (A - \theta_{i+1}) \ldots (A - \theta_2) V \]
\[ \leq U_0 + \ldots + U_d \quad \text{by Lemma} \]
\[ + V = \bigoplus_{i=0}^d U_i \]

(3) \[ \tilde{U}_i \equiv U_0 + \ldots + U_i / U_{i+1} + \ldots + U_d = V_0^\ast + \ldots + V_i^\ast / V_{i+1}^\ast + \ldots + V_d^\ast \]
\[ U_i \equiv U_0 + \ldots + U_i / U_{i+1} + \ldots + U_d = V_0^\ast + \ldots + V_i^\ast / V_{i+1}^\ast + \ldots + V_d^\ast \]

(4) \[ (A, A^\ast ; \{V_0^\ast\}_{i=0}^d, \{V_i^\ast\}_{i=0}^d) \quad \text{TD-system} \]
\[ \tilde{U}_i = (V_0^\ast + \ldots + V_i^\ast) \cap (V_{i+1}^\ast + \ldots + V_d^\ast) \]

Then \[ \text{by (3)} \]
\[ \dim \tilde{U}_i = \dim V_i^\ast = \dim V_{i+1}^\ast \]
\[ = \dim U_i \quad \dim \tilde{U}_i \]

\[ \square \]
\[ V = \bigoplus_{i=0}^{d} U_i \quad \text{w. s. decomp.} \]

\[ F_i : \quad V \rightarrow U_i \quad \text{projection} \]

\[ R = A - \sum_{i=0}^{d} \beta_i F_i \quad \text{raising map} \]

\[ \beta_i : \text{eigenvalue of } A \text{ in } V_i \]

\[ L = A^* - \sum_{i=0}^{d} \beta_i^* F_i \quad \text{lowering map} \]

\[ \beta_i^* : \text{eigenvalue of } A^* \text{ in } V_i^* \]

\[
\begin{align*}
R U_i & \leq U_{i+1}, \quad 0 \leq i \leq d, \quad U_{d+1} = 0 \\
L U_i & \leq U_{i-1}, \quad 0 \leq i \leq d, \quad U_1 = 0
\end{align*}
\]

**Proof**

\[ R|_{U_i} = A - \beta_i |_{U_i} \]

\[ L|_{U_i} = A^* - \beta_i^* |_{U_i} \]

The result follows from Lemma.
Proof of Proposition

\[ R_{d-2i}^{\ast} \mid U_{i} = (A - \alpha_{d-i} \cdot 1) \cdots (A - \alpha_{i+1} \cdot 1)(A - \alpha_{i}) \mid U_{i} \]

\[ \ker R_{d-2i}^{\ast} \mid U_{i} = U_{i} \cap (V_{d-i} + \cdots + V_{i+1} + V_{i}) \leq (V_{d}^{\ast} + \cdots + V_{i}^{\ast}) \cap (V_{d-i} + \cdots + V_{0}) \equiv 0 \]

Proof of claim

\( (A, A^{\ast}; \{ V_{d-i}^{d} \mid i = 0 \}, \{ V_{i}^{d} \mid i = 0 \}) \) T-D-system

\[ V = \bigoplus_{i=0}^{d} \widehat{U}_{i} \quad \text{w.r. decomp.} \]

where \[ \widehat{U}_{i} = (V_{d}^{\ast} + \cdots + V_{i}^{\ast}) \cap (V_{d-i} + \cdots + V_{0}) \]

Then

\[ V_{d}^{\ast} + \cdots + V_{i}^{\ast} = \widehat{U}_{i} + \cdots + \widehat{U}_{d} \]

\[ V_{d-i} + \cdots + V_{0} = \widehat{U}_{i} + \cdots + \widehat{U}_{d} \]

\[ (V_{d}^{\ast} + \cdots + V_{i}^{\ast}) \cap (V_{d-i} + \cdots + V_{0}) \]

\[ = (\widehat{U}_{0} + \cdots + \widehat{U}_{i}) \cap (\widehat{U}_{i+1} + \cdots + \widehat{U}_{d}) = 0 \]

So \[ \ker R_{d-2i}^{\ast} \mid U_{i} = 0. \]

The mapping is injective.

Since \( \dim U_{i} = \dim U_{d-i} \), the surjectivity follows from the injectivity.

\[ (A^{\ast} - \beta_{d-i}^{\ast}) \cdots (A^{\ast} - \beta_{i+1}^{\ast}) \mid U_{d-i} \]

\[ \ker L_{d-2i}^{\ast} \mid U_{d-i} = U_{d-i} \cap (V_{i+1}^{\ast} + \cdots + V_{d}^{\ast}) \leq (V_{d-i} + \cdots + V_{d}) \cap (V_{i+1} + \cdots + V_{d}) \equiv 0 \]

Proof of claim

\( (A, A^{\ast}; \{ V_{i}^{d} \mid i = 0 \}, \{ V_{i}^{d} \mid i = 0 \}) \) T-D-system

\[ \widehat{U}_{i} = (V_{d}^{\ast} + \cdots + V_{i}^{\ast}) \cap (V_{d-i} + \cdots + V_{d}) \]

\[ V = \bigoplus_{i=0}^{d} \widehat{U}_{i} \]

\[ V_{d}^{\ast} + \cdots + V_{i}^{\ast} = \widehat{U}_{i} + \cdots + \widehat{U}_{d} \]

\[ V_{d-i} + \cdots + V_{d} = \widehat{U}_{i} + \cdots + \widehat{U}_{d} \]

\[ (V_{d-i} + \cdots + V_{d}) \cap (V_{i+1}^{\ast} + \cdots + V_{d}^{\ast}) \]

\[ = (\widehat{U}_{i+1} + \cdots + \widehat{U}_{d}) \cap (\widehat{U}_{d} + \cdots + \widehat{U}_{d-i+1}) = 0 \]
\(A, A^* \in \text{End}(V)\)  
\(\text{TD-pair}\)\n\(\text{diameter } d\)

\[V = \bigoplus_{i=0}^{d} V_i^*,\quad \text{eigenspace decomposition of } A^*\]

\[E_i^* : V \rightarrow V_i^*, \quad \text{projection}\]

\[A^* = \sum_{i=0}^{d} \theta_i^* E_i^*, \quad \theta_i^* : \text{eigenvalue of } A^* \text{ on } V_i^*\]

\[E_i^* = \prod_{\nu \neq i} \frac{A^* - \theta_i^*}{\theta_i^* - \theta_\nu^*}\]

\[I = E_0^* + E_1^* + \cdots + E_d^*, \quad E_i^* E_j^* = \delta_{ij} E_i^*\]

\[A V_i^* \leq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \leq i \leq d, \quad V_d^* = V_{d+1}^* = 0\]

\[E_j^* A E_i^* = 0 \quad \text{if } |j-i| > 1\]

\[= 0 \quad \text{if } |j-i| = 1\]

\[A^* V_i^* \leq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \leq i \leq d, \quad V_1 = V_{d+1} = 0\]

\[E_j^* A^* E_i^* = 0 \quad \text{if } |j-i| > 1\]

\[= 0 \quad \text{if } |j-i| = 1\]

\[\text{Proposition} \quad l = |d - 2i|, \quad 0 \leq i \leq d\]

1. \(E_{d-i}^* A E_i^* : V_i^* \rightarrow V_{d-i}^*, \quad \text{bijection}\)

2. \(E_{d-i}^* A^* E_i : V_i \rightarrow V_{d-i}, \quad \text{bijection}\)
Proof

(1) Case \( d-i \geq i \)

Set \( W_i = V_0^* + \cdots + V_i^* = U_0 + \cdots + U_i \).

Then \( V_i^* \cong W_i / W_{i-1} \cong U_i \).

Thus \( E_i^* A_i^* E_i^* : V_i^* \rightarrow V_{i+1}^* \)

\( \cong \)

\( A : W_i / W_{i-1} \rightarrow W_{i+1} / W_i \)

\( \cong \)

\( E_{i+1} A F_i : U_i \rightarrow U_{i+1} \)

\( \cong \)

\( E_i^* A F_i : U_i \rightarrow U_{i+1} \)

Since \( F_{i-1} A^* F_i : U_i \rightarrow U_{i-1} \) is surjective,

so \( E_i^* A^* E_i^* : V_i^* \rightarrow V_{i-1}^* \).

(2) The same argument as in (1).

Corollary

(1) \( E_j^* A^* E_j^* = 0 \) \( \forall j \geq 1 \)

\( \neq 0 \) \( \forall j-i = \ell \)

(2) \( E_j A^* E_i^* = 0 \) \( \forall i < j \)

\( \neq 0 \) \( \forall i = \ell \)

Proof

(1) Case \( d-i = j-i \), \( d-i = j+i \).

\( 0 \neq E_{d-i}^* A_{d-2}^* E_i^* = E_{d-i}^* A_{d-i}^* E_i^* \)

So \( E_j A^* E_i^* \neq 0 \).

Other cases are proved in the same way.
Lemma

1. \[ E^{*}_j A^m E^*_k A^*_k A^*_i = E^{*}_j A^{m+2} E^*_i \]
   \[ \text{if} \quad k = j - m = i + 1 \quad \text{or} \quad k = j + m = i - 1 . \]

2. \[ E^{*}_j A^m E^*_k A^*_k E^*_i = E^{*}_j A^{m+2} E^*_i \]
   \[ \text{if} \quad k = j - m = i + 1 \quad \text{or} \quad k = j + m = i - 1 . \]

Proof:

1. \[ E^{*}_j A^{m+2} E^*_i = E^{*}_j A^m (E^*_i + E^*_i + \ldots + E^*_i) A^*_i E^*_i \]
   \[ = E^{*}_j A^m E^*_k A^*_k E^*_i . \]

§4 TD-pairs: TD-relations and Terwilliger's Lemma

§4.1 TD-relations

Theorem (TD-relations)

\[ A, A^* \in \text{End}(V) \quad \text{TD-pair} \quad d \]

(1) \[ \exists \beta, r, s \in \mathbb{C} \]

(TD) \[ A^3 A^* - (p+1)(A^2 A^* A^2) - A^* A^3 = s(A^2 A^* A^2) + s(A^* A A^2) . \]

Moreover, if \( d = 3 \), \( p, r, s \) are uniquely determined.

If \( d = 2 \), \( p \) is arbitrary, \( r, s \) are uniquely determined.

If \( d = 1 \), \( p, r \) are arbitrary, \( s \) is uniquely determined.

(2) \[ \exists \beta, \tau, \delta \in \mathbb{C} \]

(TD) \[ A^3 A^* - (p+1)(A^2 A^* A^2) - A^* A^3 = \tau (A^2 A^* A^2) + \delta (A^* A A^2) . \]

Moreover, if \( d = 3 \), \( \beta, \tau, \delta \) are uniquely determined.

If \( d = 2 \), \( \beta \) is arbitrary, \( \tau, \delta \) are uniquely determined.

If \( d = 1 \), \( \beta, \tau \) are arbitrary, \( \delta \) is uniquely determined.

Remark \[ \beta = \beta^* \quad \text{holds}, \]

(proof given later)
(1) \( \langle A \rangle \subseteq \text{End}(V) \) subbly generated by\( A \)
\[
\mathcal{L} = \text{Span}\{ XY - YX | X, Y \in \langle A \rangle \} \subseteq \text{End}(V)
\]

Then
\[
\mathcal{L} = \text{Span}\{ E_i^* A^* E_j - E_i A^* E_j^* | 0 \leq i, j \leq d \}
\]
\[
= \text{Span}\{ E_i^* A^* E_{i+1} - E_i A^* E_{i+1}^* | 0 \leq i \leq d - 1 \}
\]
by \( E_j^* A^* E_i = 0 \), \( i > j + 1 \).

In particular,
\[
\dim \mathcal{L} \leq d.
\]

Claim \( \{ A^*, A^* - A^* A^2 | 1 \leq i \leq d \} \) linearly independent

Proof of claim
\[
\sum_{i=0}^{d} c_i (A^* A^2 - A^* A^2) = 0, \quad c_i \neq 0.
\]
\[
E_r^* \left( \sum_{i=0}^{d} c_i (A^* A^2 - A^* A^2) \right) E_0^* = c_r (q^* - q^* E_r^*) E_r^* A^* E_0^* \neq 0
\]

Suppose \( r = 2 \).
\[
E_2^* (A^* A^2 - A^* A^2) E_0^* = 0 \quad \text{since} \quad A^* = d \sum_{i=0}^{d} q_i^* E_i^*.
\]
\[
E_2^* \left( \sum_{i=0}^{d} c_i (A^* A^2 - A^* A^2) \right) E_0^* = c_r (q^* - q^* E_r^*) E_r^* A^* E_0^* \neq 0
\]

Suppose \( r > 2 \).
\[
E_3^* (A^* A^2 - A^* A^2) E_0^* = (q^* - q^* E_2^*) E_2^* A^* E_0^* \neq 0
\]
\[
E_3^* \left( \sum_{i=0}^{d} c_i (A^* A^2 - A^* A^2) \right) E_0^* = 0
\]

(2) The same argument is available. \( \square \)
4.2 Eigenvalues

Theorem

\[(A, A^*; \{V_i\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d})\] TD-system

\(\theta_i: \) eigenvalue of \(A\) on \(V_i\), \(0 \leq i \leq d\)

\(\theta_i^*: \) eigenvalue of \(A^*\) on \(V_i^*\), \(0 \leq i \leq d\)

\(p, r, s, p^*, r^*, s^*:\) the parameters of the TD-relations

Then

\((i)\) \[p = p^* = \frac{\theta_{i+1} - \theta_i + \theta_{i-1} - \theta_{i-2}}{\theta_i - \theta_{i-1}}\]

\[= \frac{\theta_i^* - \theta_i^* + \theta_i^* - \theta_i^*}{\theta_i^* - \theta_i^*}, \quad 2 \leq i \leq d-1.\]

\((ii)\) \[Y = \theta_{i+1} - p \theta_i + \theta_{i-1}, \quad 1 \leq i \leq d-1,\]

\[y^* = \theta_i^* - p \theta_i^* + \theta_{i-1}^*, \quad 1 \leq i \leq d-1.\]

\((iii)\) \[\delta = \theta_{i+1}^2 - p \theta_{i+1} \theta_i + \theta_i^2 - r(\theta_{i+1} + \theta_i), \quad 0 \leq i \leq d-1,\]

\[\delta^* = \theta_i^2 - p \theta_i \theta_i^* + \theta_i^* - r^*(\theta_i + \theta_i^*), \quad 0 \leq i \leq d-1.\]

Remark

(1) If \(d \leq 2\), \(p, p^*\) are arbitrary. So we set \(p = p^*\).

(2) \[\{x_i\}_{i \in \mathbb{Z}}:\] \(p\)-sequence

\[\{x_i\}_{i \in \mathbb{Z}}\]

\[Y = x_{i+1} - x_i + x_{i-1} - x_{i-2}, \quad i \in \mathbb{Z},\]

\((p, r)\)-sequence

\[Y = x_{i+1} - p x_i + x_{i-1}, \quad i \in \mathbb{Z},\]

\((p, r, s)\)-sequence

\[Y = x_{i+1} - p x_i x_i + x_{i-1} - r(x_{i+1} + x_i), \quad i \in \mathbb{Z}.\]

\(p\)-seq. \(\iff\) \((p, r)\)-seq.

\[Y = x_{i+1} - p x_i + x_{i-1}\]

\[\iff Y = x_{i+1} - p x_i + x_{i-1} + x_{i-2}\]

\((p, r)\)-seq. \(\implies\) \((p, r, s)\)-seq.

\[Y = x_{i+1} - p x_i x_i + x_{i-1} - r(x_{i+1} + x_i)\]

\[(p, r)\)-seq. \(\iff\) \((p, r, s)\)-seq.

\[<\iff\]

\[x_{i+1} \neq x_i, \quad i \in \mathbb{Z}\]

\[\delta = x_{i+1}^2 - p x_{i+1} x_i + x_i^2 - r(x_{i+1} + x_i)\]

\[\delta = x_{i+1}^2 - p x_i x_i + x_{i-1}^2 - r(x_i + x_{i-1})\]

\[0 = x_{i+1}^2 - x_{i-1}^2 - p x_i(x_{i+1} - x_i) - r(x_{i+1} x_i)\]
(3) \( \{ \beta_i \}_{i=0}^k \) is extended to \( \{ \beta_i \}_{i \in \mathbb{Z}} \) as a \( \beta \)-seq.

\( \{ \beta_i \}_{i=0}^k \) is extended to \( \{ \beta_i \}_{i \in \mathbb{Z}} \) as a \( \beta \)-seq.

(4) \( \{ x_i \}_{i \in \mathbb{Z}} \) is a \( \beta \)-sequence

- type I \( q \neq 1 \)
  \[ x_i = a + b \gamma^i + c \gamma^i, \quad i \in \mathbb{Z} \]

- type II \( q = 1 \)
  \[ x_i = a + b i + c i^2, \quad i \in \mathbb{Z} \]

- type III \( q = -1 \)
  \[ x_i = a + b (-1)^i + c (-1)^i i, \quad i \in \mathbb{Z} \]

---

**Proof of Theorem**

**Notations as in §3.**

**TD-Relation**


\[ E_{i+1}(LHS \not\in TD) E_i = \left( \beta_i - \delta \delta_i \right) E_{i+1} A_i E_i \]

\[ E_{i+1}(RHS \not\in TD) E_i = \left( \beta_i - \delta \delta_i \right) E_{i+1} A_i E_i \]

\[ \delta \delta_i \not= 0, \quad E_{i+1} A_i E_i \not= 0 \]

\[ S_0 \]

\[ \delta \delta_i + \delta \delta_i \not= 0 \]

\[ (\beta, \gamma, \delta) - \text{seq.} \implies (\beta, \gamma, \delta) - \text{seq.} \implies (\beta, \gamma, \delta) - \text{seq.} \]

\[ (\beta, \gamma, \delta) - \text{seq.} \]

\[ E_{i+1}^\text{LHS} \not\in TD E_i^\text{LHS} = \left( \beta_i - \delta \delta_i \right) E^\text{LHS}_{i+1} A^\text{LHS}_i E_i \]

\[ E_{i+1}^\text{RHS} \not\in TD E_i^\text{RHS} = 0 \]

\[ E_{i+1}^\text{LHS} A^\text{LHS}_i E_i^\text{LHS} = 0 \]

\[ S_0 \]

\[ \beta_i - \delta \delta_i \not= 0 \]

\[ \{ \beta_i \}_{i \in \mathbb{Z}} \text{ is a } \beta \text{-sequence.} \]

\( \beta = \beta^* \)
§ 4.3  TD-relations revisited

pre TD-pair: \( A, A^* \in \text{End}(V) \)

\[
\begin{align*}
V &= \bigoplus_{i=0}^{d} U_i \\
F_i &: V \rightarrow U_i \quad \text{projection, } 0 \leq i \leq d \\
R, L &\in \text{End}(V) \\
R U_i &\leq U_{i+1}, \ 0 \leq i \leq d, \ U_{d+1} = 0 \\
L U_i &\leq U_{i-1}, \ 0 \leq i \leq d, \ U = 0 \\
\theta_0, \theta_1, \ldots, \theta_d &\in \mathbb{C}, \ \theta_i \neq \theta_j \\
\theta_0^*, \theta_1^*, \ldots, \theta_d^* &\in \mathbb{C}, \ \theta_i^* \neq \theta_j^* \\
\end{align*}
\]

\[
\begin{align*}
A &= R + \sum_{i=0}^{d} \theta_i F_i \\
A^* &= L + \sum_{i=0}^{d} \theta_i^* F_i \\
\end{align*}
\]

pre TD-pair

Then a pre TD-pair \( A, A^* \in \text{End}(V) \) are diagonalizable.

\[
V = \bigoplus_{i=0}^{d} V_i : \text{eigenspace decomposition of } A \\
A|_{V_i} = \theta_i : \text{eigenvalue on } V_i \\
V = \bigoplus_{i=0}^{d} V_i^* : \text{eigenspace decomposition of } A^* \\
A^*|_{V_i} = \theta_i^* : \text{eigenvalue on } V_i^* \\
\]

and

\[
\begin{align*}
U_0 + \ldots + U_i &= V_0^* + \ldots + V_i^*, \ 0 \leq i \leq d \\
U_i + \ldots + U_d &= V_i + \ldots + V_d, \ 0 \leq i \leq d \\
U_i &= (V_0^* + \ldots + V_i^*) \cap (V_i + \ldots + V_d), \ 0 \leq i \leq d \\
\end{align*}
\]

\[
(A, A^*; \{ V_i \}_{i=0}^{d}, \{ V_i^* \}_{i=0}^{d} ) \quad \text{pre TD-system}
\]
Theorem

\[(A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)\]  \(\text{p.m. TD-system}\)

\[A = R + \sum_{i=0}^d \beta_i F_i,\]
\[A^* = L + \sum_{i=0}^d \beta_i^* F_i.\]

Assume \(\{\beta_i\}_{i \in \mathbb{Z}}\) is a \((p, r, \delta)\)-sequence,
\(\{\beta_i^*\}_{i \in \mathbb{Z}}\) is a \((p, r^*, \delta^*)\)-sequence.

Set
\[\alpha_i = (p+i) \left( \beta_i \delta_i - \delta_{i+2} \beta_{i+2} + (\beta_i \delta_{i+2} + \delta_{i+2} \beta_{i+2}) - (\beta_i^* \delta_{i+2} + \delta_{i+2}^* \beta_{i+2}^*) \right)\]

Then

(i) \(A_i^3 A_i - (p+i) (A_i^2 A_i - AA_i A_i) - A_i^3 = X_1 + X_2 + X_3 + X_4 + X_5\)

\[X_1 = R^2 F - F^* R^3 - (p+i) (R^2 F^* R - RF^* R^2)\]
\[X_2 = R^2 L - L R^3 + (R^2 F^* + RF^* F^* R^3 - F^* (R^2 F^* + RF^* F^* R)\]
\[X_3 = (-p+i) (R^2 L - LR^3)\]
\[X_4 = - (p+i) \{ R^2 F^* + (RF^* F + FR^* R^3 - RF^* F^* R)\}

(ii) \(A_i^3 A_i - (p+i) (A_i^2 A_i - AA_i A_i) - A_i^3 = X_1 + X_2 + X_3 + X_4 + X_5\)

\[X_1 = (R^2 F^* + RF^* F) L - L (R^2 F^* + RF^* F)\]
\[X_2 = - (p+i) \{ R^2 F^* + RF^* F \}
\[X_3 = (-p+i) \{ R^2 F^* + RF^* F \}

Proof

\[A = R + F, \quad F = \sum_{i=0}^d \beta_i F_i, \]
\[A^* = L + F^*, \quad F^* = \sum_{i=0}^d \beta_i^* F_i.\]

(i)
\[A_i^3 A_i = (p+i) (A_i^2 A_i - AA_i A_i) - A_i^3 = X_1 + X_2 + X_3 + X_4 + X_5\]

\[X_1 = R^2 F - F^* R^3 - (p+i) (R^2 F^* R - RF^* R^2)\]
\[X_2 = R^2 L - L R^3 + (R^2 F^* + RF^* F^* R^3 - F^* (R^2 F^* + RF^* F^* R)\]
\[X_3 = (-p+i) (R^2 L - LR^3)\]
\[X_4 = - (p+i) \{ R^2 F^* + (RF^* F + FR^* R^3 - RF^* F^* R)\}

(ii)
\[A_i^3 A_i = (p+i) (A_i^2 A_i - AA_i A_i) - A_i^3 = X_1 + X_2 + X_3 + X_4 + X_5\]

\[X_1 = (R^2 F^* + RF^* F) L - L (R^2 F^* + RF^* F)\]
\[X_2 = - (p+i) \{ R^2 F^* + RF^* F \}
\[X_3 = (-p+i) \{ R^2 F^* + RF^* F \}

\[X_4 = (L^2 L + LF^* - LF + FL (RF^* F + R^2 F^* R) - RF^* F^* R)\]
\[X_5 = F^* F^* F - F^* F^* F\]
\( Y(\mathbf{A}^a - \mathbf{A}^3) + \delta(\mathbf{A}^a - \mathbf{A}^3) = Y_2 + Y_1 + Y_0 + Y_{-1} \)

\[
Y_2 = Y \left( R^2 F^r - F^b R^3 \right) \\
Y_1 = Y \left( R^2 L - L^2 R^3 + \delta \left( (RF + FR) F^r - F^b (RF + FR) \right) \right) \\
Y_0 = Y \left( (RF + FR) L - L (RF + FR) \right) + Y \left( F^r \mathbf{F}^r - F^b \mathbf{F}^r \right) \\
Y_{-1} = Y \left( F^r L - L F^r \right) + \delta (FL - LF)
\]

on \( \mathbf{U}_i \)

\( X_3 = 0 \) ok by \( \{ \alpha_{ij} \}_{i,j = 2} \) \( \beta \)-sequence

\( X_2 = Y_2 \) \( \Leftrightarrow \) \( R^2 L - (\mathbf{A}^a R - R L^2) - L R^2 = \alpha_{ij} R^2 \)

\( X_1 = Y_1 \) ok by \( \{ \beta_{ij} \}_{i,j = 2} \) \( \beta \)-sequence

\( X_0 = Y_0 \) ok by \( \{ \beta_{ij} \}_{i,j = 2} \) \( \beta \)-sequence

\( X_{-1} = Y_{-1} \) ok by \( \{ \beta_{ij} \}_{i,j = 2} \) \( \beta \)-sequence

(2) by the same argument.

**Theorem**

\( (A, \mathbf{A}^r ; \{ v_i \}_{i = 0}^d, \{ \beta_i \}_{i = 0}^d ) \) pre TD-system

\( A = R + \sum_{i = 0}^d \delta_i \mathbf{F}^r_i \)

\( A^r = L + \sum_{i = 0}^d \delta_i \mathbf{F}^r_i \)

Assume \( \{ \beta_{ij} \}_{i,j = 2} \) is a \( (p, r, \delta) \)-sequence,

\( \{ \beta_{ij} \}_{i,j = 2} \) is a \( (p, r, \delta) \)-sequence.

Set \( \rho = q + q^d \).

The

1. \( (\text{TD}) \quad A^a \mathbf{A}^r - (\mathbf{A}^a)^2 - AA^a = Y(A^a - AA^a) + \delta(A^a - AA^a) \)

\[ \Rightarrow A^a V_i \leq V_{i-1} + V_i + V_{i+1}, \quad 0 \leq i \leq 1, \]

\[ V_{d+1} = 0, \quad V_{-1} = \begin{cases} V_0 & \text{if } \beta_{d+1} = 1, \quad \beta \neq 1 \\ 0 & \text{otherwise} \end{cases} \]

2. \( (\text{TD}) \quad A^a \mathbf{A}^r - (\mathbf{A}^a)^2 - AA^a = Y(A^a - AA^a) + \delta(A^a - AA^a) \)

\[ \Rightarrow A^a V^e_i \leq V^e_{i-1} + V^e_i + V^e_{i+1}, \quad 0 \leq i \leq 1, \]

\[ V^e_{-1} = 0, \quad V^e_{d+1} = \begin{cases} V^e_0 & \text{if } \beta_{d+1} = 1, \quad \beta \neq 1 \\ 0 & \text{otherwise} \end{cases} \]
Proof

(1) \( V_i \cong V \wedge \)

\[
\begin{align*}
&= (A - \theta_i) (A^* - p \theta_i A + \theta_i^2) A^* V
\end{align*}
\]

\[
\begin{align*}
(\delta (A^* A - A^* A) + \delta (AA^* - A^* A)) V_i &= (\delta (A^* A - A^* A) + \delta (AA^* - A^* A)) A^* V \\
&= (A - \theta_i) (r(A + \theta_i) + s) A^* V
\end{align*}
\]

\[
O = (A - \theta_i) (A^* - (p \theta_i + r) A + \theta_i^2 - r \theta_i - s) A^* V
\
\begin{align*}
p \theta_i + r &= p \theta_i + (\theta_{i-1} - p \theta_i + \theta_{i-1}) = \theta_{i+1} + \theta_{i-1} \\
\theta_i^2 - r \theta_i - s &= \theta_i^2 - r \theta_i - (\theta_{i-1} - p \theta_i + \theta_i^2 - r \theta_{i-1} + \theta_{i-1}) \\
&= \theta_{i+1} (r - \theta_{i+1} + p \theta_i) = \theta_{i+1} \theta_{i-1}
\end{align*}
\]

So \( O = (A - \theta_{i+1})(A - \theta_i) A^* V \)

\[
\begin{align*}
&0 = (A - \theta_i) A^* V_i \\
&= V_{i-1} + V_i + V_{i+1}
\end{align*}
\]

\([
\begin{align*}
&i = 0, \quad A^* V_0 = V_i + V_{i+1} \\
&i = d, \quad A^* V_d = V_{i-1} + V_i
\end{align*}
\]

where \( V_1 = \{ v \in V \mid A v = \theta_{i+1} v \} \).

\( \theta_i \) is an eigenvalue of \( A \) \( \iff \) \( \theta_i, i \neq 0 \)

\[
\begin{align*}
&\text{Type I} \quad \theta_i = a + b \theta^i + c \theta^i \quad \text{and} \quad \theta_i \neq \theta_j \quad \text{if} \quad i, j \in \{0, 1, \ldots, d\}
\end{align*}
\]

\[\begin{align*}
&\text{Type II} \quad \theta_i = a + b i + c \theta^i \quad \text{and} \quad \theta_i \neq \theta_j \quad \text{if} \quad i, j \in \{0, 1, \ldots, d\}
\end{align*}\]

§4.4 Terwilliger's Lemma

\( A, A^* \in End(V) \) diagonalizable

\[
V = \bigoplus_{i=0} V_i \quad \text{eigenspace decomposition of} \quad A
\]

\[
A|_{V_i} = \theta_i \quad \text{eigenvalue of} \quad A \text{ on} \quad V_i
\]

\[
V = \bigoplus_{i=0} V_i^* \quad \text{eigenspace decomposition of} \quad A^*
\]

\[
A|_{V_i}^* = \theta_i^* \quad \text{eigenvalue of} \quad A^* \text{ on} \quad V_i^*
\]

\( E_i : V = \bigoplus_{i=0} V_i \longrightarrow V_i \quad \text{projection} \)

\( E_i^* : V = \bigoplus_{i=0} V_i^* \longrightarrow V_i^* \quad \text{projection} \)

If \( (A, A^*) \mid \{ V_i \}_{i=0}^d \mid \{ V_i^* \}_{i=0}^d \) is a TD-system,

\[
A V_i^* \leq V_i^* + V_i + V_{i+1}, \quad 0 \leq i \leq d, \quad V_i^* = V_{i+1} = 0
\]

\[
A^* V_i \leq V_i + V_i + V_{i+1}, \quad 0 \leq i \leq d, \quad V_i = V_{i+1} = 0.
\]

If \( (A, A^*) \mid \{ V_i \}_{i=0}^d \mid \{ V_i^* \}_{i=0}^d \) is a pre TD-system,

\[
A V_i^* \leq V_i^* + \ldots + V_d, \quad 0 \leq i \leq d, \quad V_d = 0
\]

\[
A^* V_i \leq V_i + \ldots + V_d, \quad 0 \leq i \leq d, \quad V_0 = 0.
\]
Terwilliger's Lemma

1. \( A(V_{k}^* + \cdots + V_d^*) \leq V_i^* + \cdots + V_d^* \)
   \[ \iff \quad E_0^* (A - AE_0^*)(A^* - B_i^*) = 0 \]

2. \( A^*(V_0 + \cdots + V_{d+1}) \leq V_0 + \cdots + V_{d+1} \)
   \[ \iff \quad E_d^* (A^* - A^* E_d^*)(A - B_{d+1}^*) = 0 \]

Proof:

1. \( A(V_{k}^* + \cdots + V_d^*) \leq V_i^* + \cdots + V_d^* \)
   \[ \iff \quad E_0^* A V_i^* = 0, \quad 2 \leq i \leq d \]
   \[ \iff \quad E_0^* A E_i^* = 0, \quad 2 \leq i \leq d \]
   \[ \iff \quad E_0^* (A - AE_0^*)(A^* - B_i^*) = 0 \]

§5 Classification of \( L \) - pairs

\((A, A^*) \{ V_{i}^{d} \}_{i=0}^{d}, \{ V_{i}^{d} \}_{i=0}^{d} \) \( L \)-system

\( V = \bigoplus_{i=0}^{d} V_{i}, \quad \dim V_{i} = 1, \quad 0 \leq i \leq d \)

\( E_i^* : \text{eigenvalue of } A \text{ on } V_{i} \)

\( V = \bigoplus_{i=0}^{d} V_{i}^*, \quad \dim V_{i}^* = 1, \quad 0 \leq i \leq d \)

\( E_i^* : \text{eigenvalue of } A^* \text{ on } V_{i}^* \)

\((1)\) \( A V_{i}^* \leq V_{i+1}^* + V_{i+1}^* + V_{i+1}^* \), \( 0 \leq i \leq d \), \( V_{d+1}^* = V_{d+1}^* = 0 \)

\( A^* V_{i} \leq V_{i-1} + V_{i} + V_{i+1} \), \( 0 \leq i \leq d \), \( V_{-1} = V_{d+1} = 0 \)

\((2)\) \( V \text{ is irreducible as an } (A, A^*) \text{-module} \)
\( L \)-system

\[
\Rightarrow
\]
\[
\text{pre } L\text{-system } (A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)
\]
\[
\{\theta_i\}_{i=0}^d: (\rho, v, \delta) \text{-sequence}
\]
\[
\{\theta_i^*\}_{i=0}^d: (\rho, \rho^*, \delta^*) \text{-sequence}
\]
\[
A(V_0 + \cdots + V_d^*) \leq V_1^* + \cdots + V_d^*
\]
\[
A^*(V_0 + \cdots + V_{d-1}) \leq V_0 + \cdots + V_{d-1}
\]

Such pre-\( L \)-systems are classified by Terwilliger's Lemma.

Moreover such a pre-\( L \)-system satisfies

(i) \( AV_i \leq V_i^* + V_i + V_{i+1}, \quad o \leq i \leq d \), \( V_d = V_d^* = 0 \)

(ii) \( A^* V_i \leq V_{i-1} + V_i + V_{i+1}, \quad o \leq i \leq d, \quad V_{d+1} = 0 \)

by the converse theorem of TD-relations.

\[ 5.1 \text{ Transition matrices} \]

\[
A, A^* \in \text{End}(V) \quad \text{pre } L\text{-pair}
\]

with data \((\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\lambda_i\}_{i=0}^{d-1})\)

\[
\begin{align*}
\theta_i \neq \theta_j, & \quad i \neq j \in \{0, 1, \ldots, d\} \\
\theta_i \neq \theta_j^*, & \quad i \neq j \in \{0, 1, \ldots, d\} \\
\lambda_i \neq 0, & \quad 0 \leq i \leq d-1
\end{align*}
\]

Set \( \lambda_{-1} = \lambda_d = 0 \).

\[ V = \bigoplus_{i=0}^d U_i, \quad \dim U_i = 1, \quad 0 \leq i \leq d \]

\[ F_i: V \rightarrow U_i \text{ projection} \]

\[ R, L \in \text{End}(V) \]

\[
\begin{align*}
RU_i &= U_{i+1}, \quad 0 \leq i \leq d, \quad U_{d+1} = 0 \\
LU_i &= U_{i-1}, \quad 0 \leq i \leq d, \quad U_{-1} = 0
\end{align*}
\]

\[ A = R + \sum_{i=0}^d \theta_i F_i \]

\[ A^* = L + \sum_{i=0}^d \theta_i^* F_i \]
\[ \lambda_i = t_i \begin{bmatrix} L & R \end{bmatrix} \begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix}, \quad -1 \leq i \leq d \]

\[ U_0 \ni u_0 \neq 0 \]

\[ u_i = R^\perp u_0 \in U_i \]

\[ L u_{i+1} = \lambda_i u_i, \quad -1 \leq i \leq d \]

Then \( A, A^T \) are diagonalizable.

\[ V = \bigoplus_{i=0}^d V_i \] eigenbasis decomposition of \( A \)

\[ A |_{V_i} = \lambda_i \]

\[ E_i : V \longrightarrow V_i \] projection

\[ A = \sum_{i=0}^d \lambda_i E_i \]

\[ E_i = \bigoplus_{\nu \neq i} A - \lambda_i \]

\[ V = \bigoplus_{i=0}^d V_i^* \] eigenbasis decomposition of \( A^T \)

\[ A^T |_{V_i^*} = \lambda_i^* \]

\[ E_i^* : V \longrightarrow V_i^* \] projection

\[ A^* = \sum_{i=0}^d \lambda_i^* E_i^* \]

\[ E_i^* = \bigoplus_{\nu \neq i} A^* - \lambda_i^* \]

\[ U_0 + U_1 + \ldots + U_i = V_0^* + V_1^* + \ldots + V_i^* \]

\[ U_i + U_{i+1} + \ldots + U_d = V_i + V_{i+1} + \ldots + V_d \]

\[ U_i = (V_0^* + V_1^* + \ldots + V_i^*) \cap (V_i + V_{i+1} + \ldots + V_d) \]
\begin{align*}
U \ni u_0 \neq 0 & \quad u_i = R^\perp u_0 \ni U_i \\
u_0, u_1, \ldots, u_d & : \quad \text{basis of } V \\
U_i \ni u_i = \frac{u_i}{u_i + \ldots + u_d} & \quad \sum V_i \\
V_i & = U_i + \ldots + U_d \\
E_i & = V_i \\
\exists V_0, u_0, \ldots, u_d & : \quad \text{basis of } V \\
u_i \ni V_i & \\
F_i u_i = u_i & \\
E_i u_i = u_i & \\
\text{Lemma} & \\
(1) & \quad (u_0, u_1, \ldots, u_d) = (v_0, v_1, \ldots, v_d) C \\
C & = (c_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
c_{ij} & = \prod_{\nu \neq j} (\theta_\nu - \theta_j)^{-1}, \quad 0 < j \leq d \\
(2) & \quad (v_0, v_1, \ldots, v_d) = (u_0, u_1, \ldots, u_d) C^{-1} \\
C^{-1} & = (\tilde{c}_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\tilde{c}_{ij} & = \prod_{\nu = i+1}^d (\theta_\nu - \theta_j)^{-1}, \quad 0 < j \leq d \\
\text{Proof} & \\
(1) & \quad u_j = \sum_{i=0}^d c_{ij} v_i \\
A u_j & = \theta_j u_j + u_{j+1} = \sum_{i=0}^d (\theta_j c_{ij} + c_{ij+1}) v_i \\
A \sum_{i=0}^d c_{ij} v_i & = \sum_{i=0}^d \theta_j c_{ij} v_i \\
\theta_j c_{ij} + c_{ij+1} & = \theta_j c_{ij} \\
c_{ij} = (\theta_j - \theta_i)^{-1} c_{i+1, j} = \cdots = (\theta_j - \theta_i)^{-1} \cdots (\theta_j - \theta_{j+1})^{-1} c_{i,j} \\
(2) & \quad v_j = \sum_{i=0}^d \tilde{c}_{ij} u_i \\
A v_j & = \theta_j v_j = \sum_{i=0}^d \theta_j \tilde{c}_{ij} u_i \\
A \sum_{i=0}^d \tilde{c}_{ij} u_i & = \sum_{i=0}^d (\theta_j \tilde{c}_{ij} + \tilde{c}_{i+1,j}) u_i \\
\theta_j \tilde{c}_{ij} & = \theta_j \tilde{c}_{ij} + \tilde{c}_{i+1,j} \\
\tilde{c}_{ij} = (\theta_j - \theta_i)^{-1} \tilde{c}_{i+1,j} = \cdots = (\theta_j - \theta_i)^{-1} \cdots (\theta_j - \theta_{j+1})^{-1} \tilde{c}_{j,j} \\
\square
\[ U_i \simeq U_0 + \cdots + U_{i-1} / U_0 + \cdots + U_{i-1} \]
\[ = V_0^* + \cdots + V_i^* / V_0^* + \cdots + V_i^* \]
\[ \simeq V_i^* \]
\[ \exists v_0^*, v_1^*, \ldots, v_d^* : \text{ basis of } V \]
\[ v_i^* \in V_i^*, \quad F_i v_i^* = u_i \]
\[ E_i^* u_i = v_i^* \]

**Lemma**

1. \[ (v_0^*, v_1^*, \ldots, v_d^*) ) C^* \]
\[ C^* = \begin{pmatrix} 1 & \ast \\ \ast & \ast \end{pmatrix} = (c_{ij}^*) \]
\[ c_{ij}^* = \prod_{\nu=i}^{1} \frac{\lambda_{\nu+}(\theta_i^* - \theta_0^*)}{c_{i+}^*}, \quad 0 \leq i \leq j \leq d \]

(2) \[ (v_0^*, v_1^*, \ldots, v_d^*) ) C_{i+}^* \]
\[ C_{i+}^* = \begin{pmatrix} 1 & \ast \\ \ast & \ast \end{pmatrix} = (c_{ij}^{i+}) \]
\[ c_{ij}^{i+} = \prod_{\nu=i}^{i+1} \frac{\lambda_{\nu+}(\theta_i^* - \theta_0^*)}{c_{i+}^*}, \quad 0 \leq i \leq j \leq d \]

§ 5.2 Equations derived from Tanigawa’s Lemma

\[ (A, A^* ) \{ V_i^* \}_{i=0}^d \quad \{ V_i^* \}_{i=0}^d \quad \{ \lambda_i \}_{i=0}^d \]

**Proposition**

Set \[ \lambda_i^* = \lambda_i - (\theta_i^* - \theta_0^*) (\theta_i^* - \theta_0^*), \quad 0 \leq i \leq d \]
\[ \lambda_{i+}^* = \lambda_{i+}^* = 0 \]

(1) \[ A (V_0^* + \cdots + V_d^* ) \leq V_i^* + \cdots + V_d^* \]
\[ \Leftrightarrow \]
\[ \lambda_{i+}^* = \frac{\theta_i^* - \theta_0^*}{\theta_{i+}^* - \theta_0^*}, \quad 0 \leq i \leq d, \quad \frac{\theta_i^* - \theta_0^*}{\theta_{i+}^* - \theta_0^*} \lambda_{i+}^* = 0 \]

(2) \[ A^* (V_0 + \cdots + V_{d-1} ) \leq V_0 + \cdots + V_{d-1} \]
\[ \Leftrightarrow \]
\[ \lambda_i = \frac{\theta_i - \theta_{d-1}}{\theta_{i+}^* - \theta_0^*}, \quad 0 \leq i \leq d, \quad \frac{\theta_i - \theta_{d-1}}{\theta_{i+}^* - \theta_0^*} \lambda_{i+}^* = 0 \]
Proof

1) \[ A^* (V_0^* + \cdots + V_d^*) \leq V_1^* + \cdots + V_d^* \]

\[ \iff \]

\[ E_v^* (A - A \cdot E_v^*) (A^* - \theta_v^*) = 0 \]

by Terwilliger's Lemma

\[ E_v^* (A - \theta_v) (A^* - \theta_v^*) = 0 \]

\[ a_v = \theta_v + \frac{\lambda_v}{(\theta_v^* - \theta_v)} \]

\[ E_v^* A E_v^* = a_v \in \mathbb{C} \]

\[ E_v^* A E_v^* V_v^* = E_v^* A V_v^* (V_v^* u_v) \]

\[ = E_v^* (\theta_v u_v + u_v) \]

\[ = \theta_v V_v^* + c_{v1} V_v^* \]

\[ = (\theta_v + \frac{\lambda_v}{(\theta_v^* - \theta_v)}) V_v^* \]

\[ \text{Lemma on page 5-8} \]

\[ E_v^* (A - \theta_v) (A^* - \theta_v^*) u_v \]

\[ \frac{\lambda_v}{(\theta_v^* - \theta_v)} u_v + \lambda_{v-1} u_{v-1} \]

\[ = E_v^* \left( (\theta_v^* - \theta_v) u_{v+1} + (\theta_v - \theta_v) (\theta_v^* - \theta_v^*) + \lambda_{v-1} \right) u_v + \lambda_{v-1} (E_v^* - \theta_v) u_{v-1} = 0 \]

\[ S_v \left( (\theta_v^* - \theta_v) \right) c_{v+1} + (\theta_v - \theta_v) (\theta_v^* - \theta_v^*) + \lambda_{v-1} \right) c_{v-1} = 0 \]

\[ \lambda_{v-1} \cdots \lambda_0 \left( \frac{\theta_v^*-\theta_v}{(\theta_v^* - \theta_v)} \cdots \frac{\theta_v^*-\theta_v}{(\theta_v^* - \theta_v)} \right) \lambda_{v-1} \]

\[ = 0 \]

(2) \[ A^* (V_0 + \cdots + V_{d-1}) \leq V_0 + \cdots + V_{d-1} \]

\[ \iff \]

\[ E_d (A^* - A \cdot E_d) (A^* - \theta_d^*) = 0 \]

by Terwilliger's Lemma

\[ E_d (A^* - \theta_d^*) (A^* - \theta_d^*) = 0 \]

\[ \theta_d^* = \frac{\lambda_d+1}{(\theta_d - \theta_{d-1})} \theta_d^* \]

\[ E_d A^* E_d = \theta_d^* \in \mathbb{C} \]

\[ E_d A^* E_d u_d = E_d A^* u_d = (V_d = u_d + u_d) \]

\[ = E_d (\lambda_d u_d + u_d) \]

\[ = (\lambda_d + \theta_d^*) u_d \]

\[ = (\theta_d - \theta_{d-1}) + \theta_d^* u_d \]

\[ \text{Lemma on page 5-6} \]

\[ E_d (A^* - \theta_d^*) (A^* - \theta_d^*) u_d \]

\[ = E_d (A^* - \theta_d^*) \left( u_{d+1} + (\theta_d - \theta_{d-1}) u_d \right) \]

\[ = E_d \left( (\theta_d^* - \theta_d) u_{d+1} + (\lambda_d + (\theta_d^* - \theta_d^*) (\theta_d - \theta_{d-1}) \lambda_{d-1} u_{d-1} \right) \]

\[ = 0 \]

\[ S_d \left( (\theta_d^* - \theta_d) c_{d+1} + (\lambda_d + (\theta_d^* - \theta_d^*) (\theta_d - \theta_{d-1}) c_{d-1} + (\theta_d - \theta_{d-1}) \lambda_{d-1} c_{d-1} \right) \]

\[ = \frac{1}{(\theta_d - \theta_{d-1}) (\theta_d^* - \theta_d^*)} \left( (\theta_d^* - \theta_d) (\theta_d^* - \theta_d^*) + (\lambda_d + (\theta_d^* - \theta_d^*) (\theta_d - \theta_{d-1}) + \theta_d - \theta_{d-1} \lambda_{d-1} \right) \]

\[ = 0 \]

\[ 0 \leq i \leq d \]

\[ \frac{\theta_d - \theta_{i-1}}{\theta_d - \theta_{i+1}} \lambda_i = 0 \]
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Assume
\[ \{ \alpha_i \} \sim (\beta, r, \delta) - \text{sequence} \]
\[ \{ \alpha_i^s \} \sim (\beta, r^s, \delta^s) - \text{sequence} \]
\[ p = q + q^{-1} \]

**Type I**
\[
\begin{align*}
\alpha_i &= a + b q^i + c q^{-i}, & i &\in \mathbb{Z} \\
\alpha_i^s &= a^s + b^s q^i + c^s q^{-i}, & i &\in \mathbb{Z}
\end{align*}
\]
Ashby-Wilson parameterization,
\[
\begin{align*}
\alpha_i &= \alpha_0 + k q^{-i} (1-q^2)(1-q^{-2}) \\
\alpha_i^s &= \alpha_0^s + k^s q^{-i} (1-q^2^s)(1-q^{-2}^s)
\end{align*}
\]
case $s = 0$, $s \neq 0$ 2-Racah

**Type II**
\[
\begin{align*}
q &\rightarrow 1, & (1-q) x &\rightarrow x, & (1-q) x^s &\rightarrow x^s \\
\frac{1-s}{1-q} &\rightarrow s, & \frac{1-s^s}{1-q} &\rightarrow s^s
\end{align*}
\]

**Type III**
\[
\begin{align*}
q &\rightarrow -1, & (1+q) x &\rightarrow x, & (1+q) x^s &\rightarrow x^s \\
\frac{1-s}{1+q} &\rightarrow s, & \frac{1-s^s}{1+q} &\rightarrow s^s
\end{align*}
\]

**Lemma**
\[
\begin{align*}
\lambda_i' &= \frac{\lambda_i^s - \lambda_i}{\lambda_i^s - \lambda_0} \lambda_i^s + \lambda_i^s, & 0 &\leq i \leq d-1 \\
\lambda_i' &= \frac{\lambda_i^s - \lambda_d}{\lambda_i^s - \lambda_0} \lambda_i^s + \lambda_d^s, & 1 &\leq i \leq d, & \lambda_0 = \lambda_0^s
\end{align*}
\]
\[
\begin{align*}
\lambda_i^s &= \frac{(1-q^{d-i})(1-q^{d-i+1})}{(1-q)(1-q^d)} \lambda_0^s, & 0 &\leq i \leq d-1 \\
\lambda_i^s &= \frac{(1-q^{a})(1-q^{a+1})}{(1-q)(1-q^d)} \lambda_0^s, & 0 &\leq i \leq d-1
\end{align*}
\]

\[
\begin{align*}
\gamma_i &\rightarrow \gamma_i^s & \gamma_i &\rightarrow \gamma_i^s & \gamma_i^s &\rightarrow \gamma_i^s \\
\frac{1-s}{1-q} &\rightarrow s, & \frac{1-s^s}{1-q} &\rightarrow s^s
\end{align*}
\]

**Type III**
\[
\begin{align*}
I &\rightarrow 1, & I &\rightarrow I, & I &\rightarrow I
\end{align*}
\]
\[
\begin{align*}
\frac{1-s}{1+q} &\rightarrow s, & \frac{1-s^s}{1+q} &\rightarrow s^s
\end{align*}
\]

**Type IV**
\[
\begin{align*}
\frac{1-s}{1+q} &\rightarrow s, & \frac{1-s^s}{1+q} &\rightarrow s^s
\end{align*}
\]

**Type V**
\[
\begin{align*}
\frac{1-s}{1+q} &\rightarrow s, & \frac{1-s^s}{1+q} &\rightarrow s^s
\end{align*}
\]

**Type VI**
\[
\begin{align*}
\frac{1-s}{1+q} &\rightarrow s, & \frac{1-s^s}{1+q} &\rightarrow s^s
\end{align*}
\]

**Type VII**
\[
\begin{align*}
\frac{1-s}{1+q} &\rightarrow s, & \frac{1-s^s}{1+q} &\rightarrow s^s
\end{align*}
\]

**Type VIII**
\[
\begin{align*}
\frac{1-s}{1+q} &\rightarrow s, & \frac{1-s^s}{1+q} &\rightarrow s^s
\end{align*}
\]

**Type IX**
\[
\begin{align*}
\frac{1-s}{1+q} &\rightarrow s, & \frac{1-s^s}{1+q} &\rightarrow s^s
\end{align*}
\]

**Type X**
\[
\begin{align*}
\frac{1-s}{1+q} &\rightarrow s, & \frac{1-s^s}{1+q} &\rightarrow s^s
\end{align*}
\]
\[\S 5.3 \quad \text{Equations derived from TD-relations}\]

\[
(A, A^*; \{v_i^d\}_{i=0}^d, \{v_i^d\}_{i=0}^d) \quad \text{pre-L-system}
\]

\[
(A, A^*; \{\theta_i^d\}_{i=0}^d, \{\lambda_i^d\}_{i=0}^d) \quad \text{data}
\]

Assume

\[
\{\theta_i^d\}_{i=0}^d : (p, r, s)-\text{sequence}
\]

\[
\{\lambda_i^d\}_{i=0}^d : (p, r^*, s^*)-\text{sequence}
\]

Set

\[
\alpha_i = (p+1)
\left(\theta_i^d \cdot \theta_i^d \cdot \theta_i^d + (\theta_i^d \cdot \theta_i^d \cdot \theta_i^d) - (\theta_i^d \cdot \theta_i^d \cdot \theta_i^d)\right)
\]

**Proposition**

\[
(\text{TD}) \quad A^d A^* - (p+1)(A^d A^* - A^d A^*) - A^d A^* = r(A^d A^* - A^d A^*) + \delta(A^d A^* - A^d A^*)
\]

\[
\iff \quad \lambda_{i-1} - (p+1)(\lambda_i - \lambda_{i+1}) = \alpha_i, \quad 0 \leq i \leq d-2
\]

\[
(\text{TD})^* \quad L^d R - (p+1)(L^d R - L^d R) - L^d R = -\alpha_i L^2 \quad \text{on } U_{d+2},
\]

\[
0 \leq i \leq d-2
\]

\[
\iff \quad \lambda_{i+2} \lambda_{i+1} \lambda_i - (p+1)(\lambda_{i+2} \lambda_{i+1} \lambda_i - \lambda_{i+2} \lambda_{i+1} \lambda_i) = -\alpha_i, \quad 0 \leq i \leq d-2
\]

\[
\iff \quad \lambda_{i+2} - (p+1)(\lambda_{i+1} - \lambda_i) = -\alpha_i, \quad 0 \leq i \leq d-2
\]
Corollary

\[ A^*(V_0^* + \cdots + V_{d-1}^*) \leq V_0^* + \cdots + V_{d-1}^* \]

\[ A^*(V_0 + \cdots + V_{d-1}) \leq V_0 + \cdots + V_{d-1} \]

\[ \Rightarrow \]

\[ (TD) \quad \text{and} \quad (TD)^* \]

In particular

\[ A V_i^* \leq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \leq i \leq d, \quad V_d^* = V_{d+1}^* = 0 \]

\[ A^* V_i \leq V_{i-1} + V_i + V_{i+1}, \quad 0 \leq i \leq d, \quad V_{i-1} = V_{i+1} = 0. \]

Proof

Use the Proposition on page 5-9, the Lemma on page 5-13 and the Theorem on page 4-13.

§5.4 The irreducibility

\[ L- \text{system } \quad (A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d) \]

\[ \Rightarrow \]

\[ \text{for } L-\text{system } \quad (A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d) \]

\[ \text{data } \quad (\lambda_1, \lambda_2)^d, \{\delta_1\}_{i=0}^d, \{\delta_2\}_{i=0}^d \]

\[ \{\lambda_1\}_{i=0}^d : (p; r, s)^{-}\text{sequence} \]

\[ \{\lambda_2\}_{i=0}^d : (p, -r, s)^{-}\text{sequence} \]

\[ p = 9 + q^{-d} \]

Type I \quad \{p \neq 21\}

\[ \lambda_2 = \delta_2^* + \delta_2^* \cdot \delta_1^* \cdot (1 - q^{e_1}) \cdot (1 - q^{e_2}) \cdot (1 - q^{e_3}) \cdot (1 - q^{e_4}) \]

\[ r_1, r_2 = 5 \delta_1^* \delta_2^* \]

Type II: Limiting cases

Call such \( L \)-system \quad special.
\[(A, A^*: \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)\] special pre-L-system

\[\Rightarrow\]
\[A V_i^* \subseteq V_i + V_i^* + V_i^{*+}, \quad 0 \leq i \leq d, \quad V_i^* = V_{i+1}^* = 0\]
\[A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, \quad 0 \leq i \leq d, \quad V_{-1} = V_{d+1} = 0\]

by Prop. on page 5-9

Lemma on page 5-10

Cor on page 5-16

So L-system

\[\Rightarrow\]

\[\text{special pre-L-system}\]

The irreducibility

\[E_{i^*}: V_{i^*}^{v_{i^*}} \rightarrow V_i\]
projection

\[E_i^*: V_{i^*}^{v_{i^*}} \rightarrow V_i^*\]
projection

Proposition

\[V: \text{ irreducible as an } \langle A, A^* \rangle \text{-module}\]

\[\Leftrightarrow\]

\[E_{i^*} V_0 \neq 0\]

Proof

\[\Rightarrow\]

Suppose \(E_{i^*} V_0 = 0\).

Then \(V_0 \subseteq V_{i^*} + \cdots + V_d^*\).

Set \(W_i = (V_0 + \cdots + V_i) \cap (V_{i+1}^* + \cdots + V_d^*)\)
\[W = W_0 + W_1 + \cdots + W_{d-1}\]

Then \((A - \delta_i) W_i \subseteq W_{i-1}, \quad (A - \delta_i) W_0 = 0,\]
\[A^* - \delta_i^* W_i \subseteq W_{i+1}, \quad (A^* - \delta_i^*) W_{d-1} = 0,\]
and so \(W\) is invariant under \(A, A^*\).

If \(0 \neq V_0 = W_0 \subseteq W \subseteq V_{i^*} + \cdots + V_d^* \neq V\).

\[V \text{ is not irreducible.} \]
\[(\iff)\]

\[W \neq 0 \iff \langle A, A^* \rangle\text{-submodule}\]

\[W = \bigoplus_{i=0}^{d} W_i V_i, \quad i_0 = \min \{i \mid W_i V_i \neq 0\}\]

\[\text{CLAIM} \quad i_0 = 0\]

\[V = \bigoplus_{i=0}^{d} U_i, \quad U_0 + \cdots + U_d = V_0 + \cdots + V_d\]

\[U_i \simeq U_0 + \cdots + U_d / U_{i+1} + \cdots + U_d\]

\[= V_0 + \cdots + V_d / V_{i+1} + \cdots + V_d\]

\[\simeq V_0\]

\[\left(A - \beta_i^* \right) V_0 = \left(A - \beta_i^* \right) U_0 \mod U_{i+1} + \cdots + U_d\]

\[= L U_{i_0}, \quad \equiv U_{i_0} - 1\]

Then \[i_0 = 0 \iff \text{the minimal \(T\).} \]

\[W \supseteq V_0, \quad \text{as \(\text{dim} V_0 = 1\).}\]

\[W \supseteq \mathcal{E}_0^* V_0 \neq 0, \quad \text{as \(\text{dim} V_0 = 1\)}\]

\[
\mathcal{E}_0^* V_0 = V_0 \]

\[
W \supseteq \mathcal{U}_0 = \left(A - \beta_0^* \right) \left(A - \beta_0 \right) U_0 \]

\[W = V\]
9. - Real case 

\[ \text{type I } + s \neq 0, s^* \neq 0 \]

Set

\[ \hat{\lambda}_i = \frac{\tilde{\lambda}_i}{s^*} \frac{2i-1}{(1-s_i^2)(1-s_i^* q^{t1}) (s_i - s_i^* q^{t1}) (s_i^* - s_i q^{t1})} \]

\[ 0 \leq i \leq d-1 \]

**Theorem**

\[ \sum_{i=0}^{d} \frac{\lambda_i}{(b_i - b_i^*)(b_i^* - b_i^*)} = \frac{\lambda_0 \cdots \lambda_{d-1}}{(b_0 - b_1) \cdots (b_0^* - b_1^*)} \]

**Calculation**

\[ \text{LHS} = \sqrt{2} \left( i, i, i, i^d \right) \left( \begin{array}{ccc} s^2 & s^*_2 & s^*_1 \\ s^1 & s & i \\ i^* & i^* & i^* \end{array} \right) \]

**Proof of Theorem outline**

\[ \Rightarrow \text{RHS} \]

9. - analogue of Pfaff-Sachs hypothesis formula.

**Pfaff-Sachs hypothesis formula**

\[ \sum_{j=2}^{n} \left( \begin{array}{c} a, b, -n \cr c, c a+b-c-n \end{array} ; 1 \right) = \frac{(c-a)_n (c-a-b)_n}{(c)_n (c-a-b)_n} \]

\[ n = 0, 1, 2, \ldots \]

9. - analogue

\[ \sqrt{2} \left( \begin{array}{c} a, b, q^* n \cr c, q a*b* \end{array} ; q, q \right) = \frac{(\frac{a}{b}, q)_n (\frac{a}{b}, q)_n}{(c)_n (\frac{a}{b}, q)_n} \]

\[ n = 0, 1, 2, \ldots \]

\(
\checkmark: \text{inequality as an } (A, A^*)\text{-module}
\)

\[ \Rightarrow \hat{\lambda}_i \neq 0, \ 0 \leq i \leq d-1 \]
$A, A^* \in \text{End}(V)$  \hspace{1cm} \text{T-D-pair}

$\Rightarrow$

4 T-D systems

$(\{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ \hspace{1cm} \text{data}

$(\{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d)$ \hspace{1cm} \text{data}

$(\lambda_i)_{i=0}^{d-1}$ \hspace{1cm} \text{data}

Proof

$\lambda_{i-1} - (p+1)(\lambda_i - \lambda_{i+1}) = \alpha_i$, $0 \leq i < d-2$, $\lambda_{d-1} = \lambda_d = 0$

$\alpha_i = (p+1) \left( \delta_i \delta_i^* - \delta_{i-1} \delta_{i+1}^* + (\delta_{i-1} \delta_{i+1}^* + \delta_{i-1} \delta_{i+1}^*) - (\delta_i \delta_i^* + \delta_{i-1} \delta_{i+1}^*) \right)$

$(A, A^* : \{V_{d-i}\}_{i=0}^d, \{V_{d-i}^*\}_{i=0}^d)$

Proof

$\alpha_i$ is symmetric w.r.t. $\{\theta_i\}_{i=0}^d \leftrightarrow \{\theta^*_i\}_{i=0}^d$
§6. Classification of dual systems of orth. poly.

§6.1 Standard basis and the associated tridiagonal matrices

\[ (A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d) \quad \text{L-system} \]

\[ V_i \neq 0, \quad i = 0, 1, \ldots, d \]

\[ w_0, w_1, \ldots, w_d \quad \text{basis of } V \]

The matrices of \( A, A^* \):

\[ D = \begin{pmatrix} \theta_0 & 0 \\ 0 & \theta_d \end{pmatrix} \]

\[ B^* = \begin{pmatrix} a_0^* & b_1^* & 0 \\ a_1^* & a_2^* & \cdots \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \end{pmatrix} \in M \]

The class of matrices that are tridiagonal, irreducible, and diagonalizable.

\[ w_0, w_1, \ldots, w_d \quad \text{standard if } w_i \in V_i, \quad i = 0, 1, \ldots, d, \]

and \( B^* \in M(\theta_0^*) \) the subclass of \( M \) that has row sum \( \theta_0^* \).

\[ c_i + a_i^* + b_i = \theta_i^*, \quad i = 0, 1, \ldots, d-1 \]

\[ a_0^* + b_0 = c_d^* + a_d = \theta_0^* \]
Proposition

(1) \( \mathcal{V}_0^* \neq 0 \), \( \mathcal{V}_i^* = E_i \mathcal{V}_0^* \subseteq \mathcal{V}_i^* \), \( 0 \leq i \leq d \)

projection of \( \mathcal{V}_0^* \) onto \( \mathcal{V}_i^* \)

Then

\( \mathcal{V}_0^* \), \( \mathcal{V}_1^* \), ..., \( \mathcal{V}_d^* \) are a standard basis.

If \( \mathcal{V}_0^* \), \( \mathcal{V}_1^* \), ..., \( \mathcal{V}_d^* \) are a standard basis

then \( \exists j \in \mathbb{C}, \quad \mathcal{V}_j^* = \mathcal{V}_j \), \( 0 \leq j \leq d \).

(2) \( \mathcal{V}_0 \neq \mathcal{V}_0^* \), \( \mathcal{V}_i^* = E_i^* \mathcal{V}_0 \subseteq \mathcal{V}_i^* \), \( 0 \leq i \leq d \)

projection of \( \mathcal{V}_0 \) onto \( \mathcal{V}_i^* \)

Then

\( \mathcal{V}_0 \), \( \mathcal{V}_1 \), ..., \( \mathcal{V}_d \) are a dual standard basis.

If \( \mathcal{V}_0 \), \( \mathcal{V}_1 \), ..., \( \mathcal{V}_d \) are a dual standard basis

then \( \exists j \in \mathbb{C}, \quad \mathcal{V}_j^* = \mathcal{V}_j \), \( 0 \leq j \leq d \).

Proof

(1) \( \mathcal{V} = \bigoplus_{i=0}^{d} \mathcal{V}_i \) weight space decomposition

\( \mathcal{U}_i = (\mathcal{V}_0^* + \cdots + \mathcal{V}_i^*) \cap (\mathcal{V}_i + \cdots + \mathcal{V}_d) \)

\( \mathcal{U}_0 = \mathcal{V}_0^* \neq \mathcal{V}_0^* = \mathcal{U}_0 \), \( R^* u_0 = u_0 \in \mathcal{U}_0 \)

\( E_i = \prod_{\beta_i}^d A - \beta_i = \prod_{\beta_i}^d A - \beta_i \cdot \prod_{\beta_i}^c A - \beta_i \)

\( \mathcal{W}_i = E_i u_0 = \left( \prod_{\beta_i}^d \frac{1}{\beta_i - \beta_i} \right) \prod_{\beta_i}^d \frac{1}{\beta_i - \beta_i} u_0 \)

\( \mathcal{W}_i = \frac{1}{(\beta_i - \beta_i) \cdots (\beta_i - \beta_i)} u_0 \mod \mathcal{U}_{i+1} + \cdots + \mathcal{U}_d \)

\( \mathcal{W}_0 + \mathcal{W}_1 + \cdots + \mathcal{W}_d = \mathcal{V}_0^* \in \mathcal{V}_0^* \)
\[ A^* (w_0, w_1, \ldots, w_d) = (w_0, w_1, \ldots, w_d) B^* \]

\[ A^* w_c = b^*_c w_c + a^*_c w_{c+1} + c^*_c w_{c+1} \]

\[ A^* w^*_0 = \delta^*_0 w^*_0 = \delta^*_0 (w_0 + w_1 + \ldots + w_d) \]

\[ \sum_{c=0}^{d} A^* w^*_c \]

\[ S_0 = b^*_0 + a^*_0 + c^*_0 = \delta^*_0 \text{ the coeff. of } w_0. \]

Suppose \( w_0, w_1', \ldots, w_d' \) is a standard basis.

Then \( \exists \ T_c \in \mathbb{C}, \ w_c = T_c w_c', \)

and the matrix of \( A^* \) w.r.t. \( w_0', w_1', \ldots, w_d' \) is

\[ B^* = \begin{pmatrix} a^*_0 & b^*_0 \\ c^*_0 & d^*_0 \\ \vdots & \vdots \\ a^*_d & b^*_d \\ c^*_d & d^*_d \end{pmatrix} \]

Each row of \( B^* = 0, \ldots, 0, c_i T_{c-i} / T_1, a_i, b_i T_{c-i} / T_1, 0, \ldots, 0 \)

\[ S_0 = c_i T_{c-1} / T_1 + a_i + b_i T_{c-1} / T_1 = \delta^*_0 = c^*_0 + a^*_0 + b^*_0. \]

Inductively \( j_0 = 1, j_1 = 1, \ldots, j_{c-1} = 1, \ldots, j_d = 1 \)

\[ \left( B^* \right)^d = \begin{pmatrix} a^*_0 & b^*_0 \\ c^*_0 & d^*_0 \\ \vdots & \vdots \\ a^*_d & b^*_d \\ c^*_d & d^*_d \end{pmatrix} \]

\[ (A, A^*; \{V_i\}_{i=0}^d, \{V_i^d \}_{i=0}^d) \text{ L-system} \]

\[ \text{data } (\{a_i^d\}_{i=0}^d, \{b_i^d\}_{i=0}^d, \{c_i^d\}_{i=0}^d) \]

\[ (A, A^*; \{V_i\}_{i=0}^d, \{V_i^d \}_{i=0}^d) \text{ L-system} \]

\[ \text{reversed data } (\{a_i^d\}_{i=0}^d, \{b_i^d\}_{i=0}^d, \{c_i^d\}_{i=0}^d) \]

\[ \text{dual standard basis } w_0, w_1', \ldots, w_d' \]

\[ A (w_0, w_1', \ldots, w_d') = (w_0, w_1, \ldots, w_d) B \]

\[ B = \begin{pmatrix} a_0 & b_0 & 0 \\ c_1 & a_1 & b_1 \\ \vdots & \vdots & \vdots \\ c_d & a_d & b_d \end{pmatrix} \in \mathcal{M}(\mathbb{C}) \]

\[ \text{standard basis } w_0, w_1, \ldots, w_d \]

\[ A^* (w_0, w_1, \ldots, w_d) = (w_0, w_1, \ldots, w_d) B^* \]

\[ B^* = \begin{pmatrix} a^*_0 & b^*_0 & 0 \\ c^*_1 & a^*_1 & b^*_1 \\ \vdots & \vdots & \vdots \\ c^*_d & a^*_d & b^*_d \end{pmatrix} \in \mathcal{M}(\mathbb{C}^d) \]
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Proof

1. $C_i = \lambda_i \cdot (\frac{x}{y} - \frac{y}{x} - \frac{y}{z} - \frac{z}{y})$, $1 \leq i \leq d$.

2. $b_i = \frac{1}{\lambda_i} \cdot (x \cdot \frac{x}{y} - \frac{y}{x} - \frac{y}{z} - \frac{z}{y})$, $0 \leq i < d$.

3. $c^*_i = \lambda_i \cdot \frac{1}{\lambda_i} \cdot (x \cdot \frac{x}{y} - \frac{y}{x} - \frac{y}{z} - \frac{z}{y})$, $1 \leq i \leq d$.

4. $b_c = \frac{1}{\lambda_c} \cdot (x \cdot \frac{x}{y} - \frac{y}{x} - \frac{y}{z} - \frac{z}{y})$, $0 \leq c < d$.

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Theorem

1. $A = \mathbb{R} \times \mathbb{R}$, $A_0 = \mathbb{R} \times \mathbb{R}$, $A_1 = \mathbb{R} \times \mathbb{R}$.

2. $R_0 = \mathbb{R} \times \mathbb{R}$, $R_1 = \mathbb{R} \times \mathbb{R}$, $R_2 = \mathbb{R} \times \mathbb{R}$.

3. $L_0 = \mathbb{R} \times \mathbb{R}$, $L_1 = \mathbb{R} \times \mathbb{R}$, $L_2 = \mathbb{R} \times \mathbb{R}$.

4. $V = \mathbb{R} \times \mathbb{R}$, $V_0 = \mathbb{R} \times \mathbb{R}$, $V_1 = \mathbb{R} \times \mathbb{R}$, $V_2 = \mathbb{R} \times \mathbb{R}$.
\[ A \omega_i = b_{\omega_i} w_{\omega_i} + a_{\omega_i} \omega_i + c_{\omega_i} w_{\omega_i} \]

\[ \text{mod } \hat{D}_0 + \cdots + \hat{D}_i \]

\[ \text{RHS} = c_{\omega_i+1} \frac{\hat{\lambda}_{\omega_i} \cdots \hat{\lambda}_{\omega_i-i}}{(\hat{\beta}^*_d - \hat{\beta}^*_i) - (\hat{\beta}^*_d - \hat{\beta}^*_i)} \hat{Q}_{\omega_i} \]

\[ \text{LHS} \]

\[ A = \hat{R} + \sum_{\omega_i} \hat{Q}_{\omega_i} \hat{F}_{\omega_i} \]

\[ A \hat{\omega}_i = \hat{Q}_{\omega_i} \text{ mod } \hat{D}_i \]

\[ A(\hat{D}_0 + \cdots + \hat{D}_i) \subseteq \hat{D}_0 + \cdots + \hat{D}_i \]

\[ \frac{\hat{\lambda}_i \cdots \hat{\lambda}_{i+1}}{(\hat{\beta}^*_d - \hat{\beta}^*_i) - (\hat{\beta}^*_d - \hat{\beta}^*_i)} \hat{Q}_{\omega_i} \]

\[ c_{\omega_i} = \hat{\lambda}_i \frac{(\hat{\beta}^*_d - \hat{\beta}^*_i) \cdots (\hat{\beta}^*_d - \hat{\beta}^*_i)}{(\hat{\beta}^*_d - \hat{\beta}^*_i) \cdots (\hat{\beta}^*_d - \hat{\beta}^*_i)}, \quad 0 \leq i \leq d-1 \]

(2) Reverse the ordering \( V_0^*, V_1^*, \ldots, V_d^* \)

\[ (A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d) \]

standard basis \( w_0, w_1, \ldots, w_i, \ldots, w_d \in V_i^* \)

\[ A(w_0, \ldots, w_i, w_j) = (w_0, \ldots, w_i, w_j) \begin{pmatrix} 0_{i \times i}^j \end{pmatrix} B \begin{pmatrix} 0_{i \times j} \end{pmatrix} \]

\[ \begin{pmatrix} 0_{i \times i} \end{pmatrix} B \begin{pmatrix} 0_{i \times j} \end{pmatrix} = \begin{pmatrix} a_{d} b_{d} & a_{d-1} b_{d-1} & \cdots & a_{1} b_{1} & a_{0} b_{0} \end{pmatrix} \]

\[ b_{d,i} = c_{\omega_i} = \frac{\gamma_i \cdots \gamma_{d-i}}{(\hat{\beta}^*_d - \hat{\beta}^*_i) \cdots (\hat{\beta}^*_d - \hat{\beta}^*_i)} \]

\[ (A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d) \]

\[ (A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d, \{\lambda_{d-i}\}_{i=0}^d) \]
(3) Use the L-system
\[(A^*, A; \{ V_i^* \}_{i=0}^d, \{ V_i \}_{i=0}^d)\]

If the ordering of \( V_0^*, V_1^*, \ldots, V_d^* \) is reversed,
\[(A^*, A; \{ V_i \}_{i=0}^d, \{ V_i^* \}_{i=0}^d)\] has data
\[(\{ \beta_i \}_{i=0}^d, \{ \beta_i^* \}_{i=0}^d, \{ \lambda_{d-i-1} \}_{i=0}^d).\]

The claim for \( C^* \) follows from (2).

(4) Use the L-system
\[(A^*, A; \{ V_i \}_{i=0}^d, \{ V_i^* \}_{i=0}^d).\]

The claim for \( C^* \) follows from (2),
since \((A^*, A; \{ V_i^* \}_{i=0}^d, \{ V_i \}_{i=0}^d)\) has data
\[(\{ \beta_i \}_{i=0}^d, \{ \beta_i^* \}_{i=0}^d, \{ \lambda_{d-i-1} \}_{i=0}^d).\]

\[\therefore\]

\[8.6.3 \text{ Classification of dual systems of orth. poly.} \]
\[(A, A^*; \{ V_i \}_{i=0}^d, \{ V_i^* \}_{i=0}^d)\]
L-system on \( V \cong C^{d+1} \)

\[\downarrow\]

\[D, B^* \in M(\mathfrak{s}^*) \text{ via standard basis}\]

\[\text{diag.} \quad L\text{-pair on } C^{d+1}\]

\[\cong\]

\[B \in M(\mathfrak{S}), D^* \text{ via dual standard basis}\]

\[\text{diag.} \quad L\text{-pair on } C^{d+1}\]

\[\exists S \text{ nonsingular matrix}\]
\[S^* BS = D^*\]
\[S^* B^* S = D^*\]
\[ M(\theta_0) \dpi{6} B = \begin{pmatrix} a_0 & b_0 & 0 \\ c_1 & a_1 & b_1 \\ \vdots & \ddots & \vdots \\ 0 & c_{d-1} & a_d \\ & & c_d \\ & & & b_d \end{pmatrix} \]

triangular
indivisible
diagonalizable
row sum \( \theta_0 \)

\[ \rho(\theta_0) \in \{ p_i(\theta_0) \}_{i=0}^{d} \]

orthogonal polynomials
support \( \{ \theta_0, \theta_1, \ldots, \theta_d \} \)

\[ p_0(\theta_0) = 1, \quad 0 \leq i \leq d \]

\[ \begin{array}{c}
\lambda \cdot p_i(\theta_0) + a_i \cdot p_i(\theta_1) + b_i \cdot p_{i+1}(\theta) \\
p_i(\theta_0) = 0, \quad p_i(\theta_1) = 1, \quad b_d = 1 \\
p_{d+1}(x) = \frac{1}{b_0 b_1 \cdots b_{d-1}} (x-\theta_0)(x-\theta_1) - (x-\theta_d) \\
\end{array} \]

min. poly. of \( B \)

\[ M(\theta^*) \dpi{6} B^* = \begin{pmatrix} a_0^* & b_0^* & b_1^* & \cdots & b_d^* \\ c_1^* & a_1^* & b_1^* & \cdots & b_d^* \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & c_{d-1}^* & a_d^* & \cdots & b_d^* \\ & & & c^*_d & b_d^* \end{pmatrix} \]

orthogonal polynomials
support \( \{ \theta_0^*, \theta_1^*, \ldots, \theta_d^* \} \)

\[ p_i^*(\theta_0^*) = 1, \quad 0 \leq i \leq d \]

\[ \{ p_i(\theta_0) \}_{i=0}^{d}, \quad \{ p_i(\theta_0^*) \}_{i=0}^{d} : \text{dual} \]

\[ p_i^*(\theta_0^*) = \frac{p_i^*(\theta_0^*)}{p_i^*(\theta_0^*)}, \quad \forall i, j \in \{0, 1, \ldots, d\} \]

All dual systems of orthogonal polynomials
arise in this way from L-pairs.
Theorem

(1) \[ p_i^e(x) = \sum_{\nu=0}^{i} \frac{(\lambda_{\nu} - \lambda_{\nu-1}) \cdots (\lambda_0 - \lambda_{-1})}{\lambda_{\nu} \cdots \lambda_{0}} (x - \lambda_0) \cdots (x - \lambda_{\nu-1}), \quad 0 \leq i \leq d \]

(2) \[ p_i^e(x) = \sum_{\nu=0}^{i} \frac{(\lambda_{\nu} - \lambda_{\nu-1}) \cdots (\lambda_0 - \lambda_{-1})}{\lambda_{\nu} \cdots \lambda_{0}} (x - \lambda_0) \cdots (x - \lambda_{\nu-1}), \quad 0 \leq i \leq d \]

Proof

(1) Set

\[ p_i^e(x) = \sum_{\nu=0}^{i} c_{nu} (x - \lambda_0) \cdots (x - \lambda_{\nu-1}), \quad 0 \leq i \leq d \]

Then

\[ A \cup \{ V_i \}_{i=0}^{d}, \quad V = \bigoplus_{i=0}^{d} U_i \]

Weight space decomposition

\[ U_i = (V_0^e + \cdots + V_i^e) \cap (V_{i+1} + \cdots + V_d) \]

Projection

\[ F_i : V \longrightarrow U_i \]

\[ A = R + \sum_{\nu=0}^{d} c_{nu} F_i, \quad R U_i = U_{i \uparrow}, \quad 0 \leq i \leq d, \quad U_{d \downarrow} = 0 \]

\[ A^* = L + \sum_{\nu=0}^{d} c_{nu} F_i, \quad L U_i = U_{i \downarrow}, \quad 0 \leq i \leq d, \quad U_{d \uparrow} = 0 \]

\[ U_0 \neq U_0 \neq 1, \]

\[ U_i \ni u_i = R^* u_i = (A - \lambda_0) \cdots (A - \lambda_{i-1}) u_i \]

\[ p_i^e(A) u_0 = \sum_{\nu=0}^{i} t_{nu} u_{\nu} \]

Claim

\[ p_i^e(A) u_0 \in V_i \]

Proof

\[ k_0 = 1, \quad k_{i} = \frac{b_0 b_1 \cdots b_{i-1}}{c_0 c_2 \cdots c_i} = \frac{b_i}{c_i} k_{i-1} \]

Set

\[ w_i = k_i p_i^e(A) u_0. \]

This means

\[ w_0, w_1, \ldots, w_d \]

are a dual standard basis.

In particular, \[ w_i \in V_i. \]
$V_i^x \in P_i(A) u_0 = \sum_{\nu=0}^{\infty} t_{\nu i} u_{\nu}$

By Lemma on page 5-8

$v_i^x = \sum_{\nu=0}^{\infty} c_{\nu i} u_{\nu} \in V_i^x$

$c_{\nu i} = \frac{\lambda_{\nu+1} \ldots \lambda_{\nu+1}}{(\theta_{\nu}^x - \theta_0^x) \ldots (\theta_{\nu}^x - \theta_m^x)}$

On the other hand,

$t_{0 i} = t_i^y (\beta_0) = 1$

So

$t_{\nu i} = \frac{c_{\nu i}}{c_{\nu i}} = \frac{(\theta_{\nu}^x - \theta_0^x) \ldots (\theta_{\nu}^x - \theta_m^x)}{\lambda_0 \ldots \lambda_{\nu-1}}$

$\text{Type I: } q \neq 1$

$\beta_i = \beta_0 + c_i \frac{1}{q^x} (1-q^x)(1-s^x \gamma_i^x), \quad 0 \leq i \leq d$

$\beta_i^* = \beta_0^* + c_i \frac{1}{q^x} (1-q^x)(1-s^x \gamma_i^x), \quad 0 \leq i \leq d$

$\lambda_i = \frac{q^x}{f} \frac{1}{q^x} (1-q^x)(1-s^x \gamma_i^x)(1-s^x \gamma_i^{x+1})(1-s^x \gamma_i^{x+1})$

$r_i r_i = ss^x q^{d+1}$

Theorem

(1) $P_2(x) = \frac{1}{4d} \left( q^x, s^x \phi_i^x, q^x \gamma_i^{x+1}, s^x \gamma_i^{x+1} \right), \quad 0 \leq i \leq d$

$\chi = \frac{1}{s^x} \left( \gamma_i^{x+1} = \beta_0 + c_i \frac{1}{q^x} (1-q^x)(1-s^x \gamma_i^{x+1}) \right)$

$b_i = \frac{k_i (1-q^x)(1-s^x \gamma_i^{x+1})(1-s^x \gamma_i^{x+1})(1-s^x \gamma_i^{x+1})}{(1-s^x \gamma_i^{x+1})(1-s^x \gamma_i^{x+1})}, \quad 0 \leq i \leq d$

$c_i = \frac{k_i}{s^x} \frac{(1-q^x)(1-s^x \gamma_i^{x+1})(1-s^x \gamma_i^{x+1})}{(1-s^x \gamma_i^{x+1})(1-s^x \gamma_i^{x+1})}, \quad 1 \leq i \leq d$
(2) \[ p^+_i(x) = \frac{1}{4^i} \left( \frac{q^i x}{q^i x^i}, \frac{q^i y}{q^i y^i}, \frac{q^i x^i y}{q^i x^i y^i}, \frac{q^i y^i}{q^i y^i} \right), \quad 0 \leq i \leq d \]

\[ x = \frac{q^i y}{q^i y^i} = \delta_{i0} + \delta_{i1} - \frac{q^i}{q^i y^i} \left( 1 - q^i y^i \right) \]

\[ b_i^s = \frac{q^{-i} (q^{-i} y^i)}{(1 - q^{-i} y^i) (1 - q^{-i/y^i}) (1 - q^{i/y^i})} \quad 0 \leq i \leq d-1 \]

\[ C_i^s = \frac{q^{i/y^i} (1 - q^i y^i) (1 - q^{-i/y^i}) (1 - q^{i/y^i})}{(1 - q^{i/y^i}) (1 - q^{i/y^i})} \quad 1 \leq i \leq d \]

Type II: \[ y \to 1, \quad \frac{1}{1-q} \to \frac{1}{1-q}, \quad \frac{q}{1-q} \to \frac{q}{1-q} \]

\[ \begin{align*}
1 - \frac{1}{1-q} & \to 1 - \frac{1}{1-q} \\
\frac{1}{1-q} & \to \frac{1}{1-q} \\
\frac{q}{1-q} & \to \frac{q}{1-q}
\end{align*} \]

Type III: \[ y \to 1, \quad \frac{1}{1-q} \to \frac{1}{1-q}, \quad \frac{q}{1-q} \to \frac{q}{1-q} \]

\[ \begin{align*}
1 - \frac{1}{1-q} & \to 1 - \frac{1}{1-q} \\
\frac{1}{1-q} & \to \frac{1}{1-q} \\
\frac{q}{1-q} & \to \frac{q}{1-q}
\end{align*} \]

References

§1, §2


§3, §4
Ref-2

§3. §4. cont'


§5. §6

P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other, Linear Algebra Appl., 330 (2001), 149-203.